Morse Theory: Morse-Bott Functions and Homology

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This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Contents

1 Introduction

Morse theory provides a method to study the topology of a manifold via the properties of smooth real-valued functions. In particular, the critical points of the function contain the information to determine the homotopy type, homology, and other important relationships. Morse theory is a higher dimensional analogue of critical point theory, developed initially in the 1920s by Marston Morse [1, 2]. The Morse inequalities were soon established, following work in min-max theory [3]. Morse theory was further developed with the help of Thom, Smale, Milnor, Witten and many others [4]. For example, the construction of the Smale-Witten chain complex using the theory of flow lines led to Morse homology [5]. The subject continued to expand with the study of Morse-Bott functions and Bott's work in periodicity [6]. There are now countless applications and branches of Morse theory, such as connections to dynamical systems, algebraic geometry and a number of areas in topology.

The focus of the first half of the report is the main results from Morse Theory and Morse homology. This begins with an introduction of Morse functions and their properties in Chapter 2.

Chapter 3 describes the cellular construction, and therefore its homotopy type as a CW-complex, of the manifold using the critical points of a Morse function. The resultant methods allow the homotopy type of a manifold to be immediately known from just one simple function. This is one of the most fundamental results in Morse theory, and implies that any smooth compact manifold can be expressed as a CW-complex.

The Morse inequalities are presented in Chapter 4. These describe an upper bound on the rank of the homology groups in terms of the number of critical points. The Morse inequalities describe a relationship between homology and the critical points, and indeed, it is possible to determine the homology groups directly from the properties of the critical points.

In Chapter 5, the methods for obtaining the Morse homology are discussed, along with some examples. In many cases, Morse homology can provide a very simple alternative chain complex, similar to that of cellular homology.

This concludes the study of spaces using Morse functions. Morse functions are special cases of Morse-Bott functions, which will be the subject of the remainder of the report. For a Morse-Bott function, the analogous results regarding homotopy type and homology are more complex. However, in many cases they provide an elegant alternative to the use of a Morse function.

Morse-Bott functions are introduced in Chapter 6. They are defined similarly to Morse functions, but without the condition that the critical points are isolated. The function has a set of critical submanifolds, whose properties give corresponding results to those of the critical points of a Morse function.

In Chapter 7, Lens spaces are defined. They provide a collection of manifolds on which there is a natural Morse-Bott function. The associated critical submanifolds and their properties will be calculated, and this function will be revisited in later chapters.

There are multiple approaches to calculating the homology from a Morse-Bott function, which are considered in Chapter 8. In general these methods are more complex, however a method can be chosen that best suits the situation and there are examples where a method can lead to a very simple solution. One such approach will be applied to the Morse-Bott function on the Lens space, where it is possible to determine the homology using a simple filtration.

Finally, in Chapter 9, the concept of flow categories is described. The flow between critical points is used in the construction Morse homology. An equivalent concept for Morse-Bott functions is developed in [7], which has the potential to give additional information about a manifold.

1.1 Key Definitions

First, there are a number of objects to be defined that will be used throughout the paper. For example, the main subject of study will be manifolds, maps between spaces, and also a number of topological concepts that will be used in calculations.

Definition 1.1 (Manifold). A *smooth manifold* is a space M that is:

- *Hausdorff*: $\forall x, y \in M, \exists$ open neighbourhoods $U_x \ni x, U_y \ni y$, with $U_x \cap U_y = \emptyset$,
- *Second countable*: M has a countable basis for its topology,
- *Locally Euclidean*: M has an open cover $\{U_\alpha\}$ with $U_\alpha \subset M$ open $\forall \alpha \in I$ and I is some index set. Then there is a bijection $\phi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha \subset \mathbb{R}^n$ open, satisfying the following. $\forall \alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$,

$$\phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

  is smooth.

Essentially, a smooth manifold is a space that locally resembles $\mathbb{R}^n$. [8]

Definition 1.2 (Tangent space). The *tangent space* T_pM to a smooth manifold M at the point $p \in M$ is defined as the set of derivations on the set of continuous functions $C^\infty(M, p)$ in a neighbourhood of p on M.
A derivation is a linear map $\delta : C^\infty(M, p) \to \mathbb{R}$ such that

$$\delta(f\dot{g}) = f(p)\delta(g) + \delta(f)g(p).$$

The derivations are the tangent vectors to M at p. Then T_pM is a vector space. [8]

Definition 1.3 (Riemannian metric). A *Riemannian metric* is a collection of inner products $g_p : T_pM \times T_pM \to \mathbb{R}$. For vector fields $X, Y \in T_pM$, the map $p \mapsto g_p(X_p, Y_p)$ is smooth.

Definition 1.4. A map $f : M \to N$ between two smooth manifolds is smooth if the maps

$$\phi_j^N \circ f \circ (\phi_i^M)^{-1}|_{\phi_i^M(U_i \cap f^{-1}(V_j \cap f(U_i)))}$$

are smooth for all $i \in I, j \in J$ where $\{U_i\}_{i \in I}$ is an open cover for M and $\{V_j\}_{j \in J}$ is an open cover for N, and $\phi_i^M : U_i \to \tilde{U}_i \subset \mathbb{R}^m, \phi_j^N : V_j \to \tilde{V}_j \subset \mathbb{R}^n$ the associated maps as in definition 1.1. [8]

Definition 1.5 (Diffeomorphism). A map $f : M \to N$ for M, N smooth manifolds is called a *diffeomorphism* if f is smooth, bijective, and both f, f^{-1} are differentiable.

Then we say that two differentiable manifolds M, N are *diffeomorphic* if there exists a diffeomorphism $f : M \to N$. [8]

Definition 1.6 (Homotopic maps). Let X, Y be two topological spaces, and $f, g : X \to Y$ two continuous maps. Then f is *homotopic* to g, denoted $f \simeq g$, if there exists a continuous family of continuous maps

$$h_t(x) : X \to Y$$

with $t \in [0, 1]$ such that the following holds.

$$\begin{aligned} h_0(x) &= f(x), \\ h_1(x) &= g(x), \end{aligned}$$

and $h_t(x)$ is a continuous function of x, and of t. Then, $h_t(x)$ is called a homotopy from f to g. Homotopy is an equivalence relation. [9, 10]

Definition 1.7 (Homotopy equivalence). Two spaces X, Y are *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that

$$\begin{aligned} f \circ g &\simeq id_Y \\ g \circ f &\simeq id_X. \end{aligned}$$

[11]

Definition 1.8 (Deformation retract). Let $A \subset X$. Then A is a *deformation retract* of X if there is a retraction $r : X \to A$ with $i \circ r \simeq id$. The map i is inclusion, and r is a retraction, with $r(a) = a \; \forall a \in A$. [9]

Definition 1.9 (Chain complex). A sequence of abelian groups

$$... \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to ... \xrightarrow{\partial_1} C_0 \xrightarrow{0}$$

with ∂_i homomorphisms and $\partial_i \partial_{i+1} = 0 \, \forall i$, is called a *chain complex.* [11]

Definition 1.10 (Homology). For a chain complex as above, define the n^{th} homology group as the quotient group

$$H_n(C_*) = ker \; \partial_n / im \; \partial_{n+1}.$$

[11]

Definition 1.11 (Relative homology). Let A be a subspace of X, and define the quotient chain complex $C_*(X, A) := C_*(X)/C_*(A)$. The homology groups $H_k(X, A) := H_k(C_*(X, a))$ are called the *relative homology groups.* [11]

Theorem 1.12 (Excision). *Let $Z \subset A \subset X$, and suppose the closure of Z is contained in the interior of A. Then for all n, there is an isomorphism*

$$H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A).$$

[11]

Definition 1.13 (Exact sequence). Let A_i be abelian groups, and $f_i : A_i \to A_{i-1}$ smooth maps. Then the sequence given by

$$... \xrightarrow{k+1} A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} ...$$

is called a *long exact sequence* if

$$ker \; f_i = im \; f_{i+1} \quad \forall i.$$

An exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a *short exact sequence.* In this case, we have that $C = B/Im(f)$. [12]

Definition 1.14 (Exact sequence of the pair). For the pair A, X with $A \subset X$, we have the following short exact sequence.

$$0 \to C_*(A) \xrightarrow{i} C_*(X) \xrightarrow{j} C_*(X, A) \to 0.$$

This induces the long exact sequence on homology, called the *long exact sequence of the pair*:

$$... \to H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\partial} H_{k-1}(A) \to ... \to H_0(X,A).$$

[12, 11]

Definition 1.15 (Fundamental class). Let M be a manifold and R a commutative ring. For the map $H_n(M;R) \to H_n(M, M-x;R)$, a *fundamental class* $[M]$ for M is an element of $H_n(M;R)$, with its image in $H_n(M, M-x;R)$ a generator for all $x \in M$. A fundamental class exists if M is R-orientable. [11]

Definition 1.16 (Degree of a map). Let M, N be two manifolds of the same dimension, n, and $[M], [N]$ their respective fundamental classes. Then, if $f : M \to N$ is a map, and f_k the induced map on the k^{th} homology, then the *degree* d of f is the unique integer such that

$$f_n([M]) = d[N] \in H_n(N).$$

[13]

Definition 1.17 (Covering space). A space X has a *covering space* $\tilde{X}$ if there is a *covering map* $p : \tilde{X} \to X$ such that every $x \in X$ has an open neighbourhood U_x with $p^{-1}(U_x)$ a disjoint union of open sets in $\tilde{X}$ that get mapped homeomorphically by p onto U_x.

Then U_x is evenly covered, and the number of *sheets* of the cover is equal to the number of disjoint open sets in $\tilde{X}$. [11]

2 Morse Functions

Definition 2.1 (Critical Point). For a smooth function $f : M \to \mathbb{R}, p \in M$ is a *critical point* if $f_* : T_pM \to T_{f(p)}\mathbb{R} = 0$. Equivalently, if $(x^1, x^2, ..., x^n)$ are local coordinates in a neighbourhood of p, and

$$\frac{\partial f}{\partial x^1}(p) = \frac{\partial f}{\partial x^2}(p) = ... = \frac{\partial f}{\partial x^n}(p) = 0,$$

then p is a critical point. Denote by $Cr(f)$ the set of critical points of the function f. [6]

Definition 2.2. Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M, and let p be a critical point of f. Then

- p is *non-degenerate* if its Hessian $H_{f,p}(x)$ is non-degenerate,
- p is *isolated* if $\exists \epsilon > 0$ such that q is not a critical point $\forall q \in B_\epsilon(p)$,
- The *index* λ_p of the critical point p of a smooth function $f : M \to \mathbb{R}$ is defined as the maximal dimension of the tangent space T_pM on which the Hessian $H_p(f)$ is negative definite. This can also be described as the number of negative eigenvalues of $H_{f,p}(x)$. [6]

Definition 2.3 (Morse Function). A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if its critical points are all non-degenerate. [6]

The non-degeneracy of the critical points implies that the critical points of a Morse function are isolated.

Definition 2.4 (Self-indexing). A function $f : M \to \mathbb{R}$ is called *self-indexing* if for all $p \in Cr(f)$,

$$f(p) = \lambda_p.$$

[3]

Lemma 2.5. *Let M be a compact manifold. Then there exists a self-indexing Morse function $f : M \to \mathbb{R}$. [3]*

2.1 The Morse Lemma

One of the most fundamental theorems in Morse theory is the *Morse Lemma*. It describes how a Morse function can be expressed in local coordinates, in a simple quadratic form.

Theorem 2.6 (Morse Lemma). *Let p be a non-degenerate critical point of index λ for a smooth function $f : M \to \mathbb{R}$. Then there exist local coordinates $(x^1, ..., x^n)$ in a neighbourhood U of p with*

$$x^i(p) = 0 \quad \forall i, \;\; \textit{and} \quad f = f(p) - (x^1)^2 - ... - (x^\lambda)^2 + (x^{\lambda+1})^2 + ... + (x^n)^2.$$

[6], [3]

Proof. We will prove the Morse Lemma as in [3], where L. Nicolaescu begins by first proving the following theorem.

Theorem 2.7. *Let $f : M \to \mathbb{R}$ be a smooth function on M, where $dim(M) = m$, and let p be a non-degenerate critical point of f. Then there exist coordinates $(x^1, x^2, ..., x^m)$ on a neighbourhood U of p with*

$$x^i(p) = 0 \quad \forall i, \;\; \textit{and} \quad f = f(p) + \frac{1}{2} H_{f,p}(x)$$

where $H_{f,p}(x) = \sum_{i,j} h_{ij} x^i x^j$ with $h_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}(p)$ is the Hessian of f.

Proof. We may assume that $f(p) = 0$. Then there exist coordinates (x^i) on a neighbourhood N of p s.t. $x^i(p) = 0 \;\; \forall i$, these are are fixed by a choice of diffeomorphism $\Phi : \mathbb{R}^m \to N$ with $\Phi(0) = p$.

Then we set $\psi(x) = f(\Phi(x))$ and define $\psi_t : \mathbb{R}^m \to \mathbb{R}$ by

$$\psi_t(x) = (1 - t)\psi(x) + tQ(x)$$

where Q is the quadratic form

$$Q = \frac{1}{2} \sum_{i,j} \frac{\partial^2 \psi}{\partial x^i \partial x^j}(0) x^i x^j.$$

The vector field

$$V_t(x) = \frac{d}{dt}\phi_t(x),$$

determines an open neighbourhood $U \subset \Phi^{-1}(N)$ of 0 and unique one-parameter family of embeddings ϕ_t of U into $\mathbb{R}^m$ such that for all $t \in [0, 1]$,

$$\phi_t(0) = 0, \; \textit{and} \;\; \phi_t \circ \psi_t = \Phi(x) \;\; \textit{on } U. \tag{1}$$

Now, if we differentiate the above composition with respect to t, we have

$$\begin{aligned} &\frac{\partial \phi_t}{\partial t} \circ \psi_t + (V_t \phi_t) \circ \psi_t = 0 \\ \Longleftrightarrow \; &Q - \psi = V_t \phi_t \quad \textit{on } \phi_t(U), \; \forall t \in [0, 1] \end{aligned} \tag{2}$$

using the definition of $\phi_t(x)$ in terms of $Q(x)$ and $\phi(x)$ from earlier.

If a vector field V_t can be found that satisfies $V_t(0) = 0 \ \forall t \in [0,1]$, and (2) on some neighbourhood W of 0, then

$$\mathcal{N} = \bigcap_{t \in [0,1]} \phi_t^{-1}(W)$$

is a neighbourhood of 0. In this case, ψ_t satisfies (1) on $\mathcal{N}$.

We will now prove the existence of such a V_t.

Definition 2.8 (Terminology and notation).

- Let f, g be functions in a neighbourhood of 0. Then f is *equivalent* to g if there exists a neighbourhood U of 0 with $f|_U = g|_U$.
- The equivalence class $[f]$ is called the *germ* of f at 0.
- We denote by $\mathcal{E}$ the collection of all germs of smooth functions at 0. $\mathcal{E}$ is a ring.
- The kernel of the surjective map $\mathcal{E} \to \mathbb{R}$ is a maximal ideal and is denoted by $\mathfrak{m}$. Note this map is induced by the evaluation at 0 map of f.

Lemma 2.9.

- $\mathfrak{m}$ *is generated by the germs of the coordinate functions* x^i.
- *If for* $|\alpha| < k$, $(D^\alpha f)(0) = 0$, *then the germ* $[f] \in \mathfrak{m}^k$. *This implies that* $[\psi] \in \mathfrak{m}^2$ *and* $\psi - Q \in \mathfrak{m}^3$.
- *Let* J_ψ *be the Jacobian ideal in* $\mathcal{E}$, *generated by the germs at* 0 *of* $\partial_{x^i}\psi$. *Then* $J_\psi = \mathfrak{m}$.

Now let $\delta := \psi - Q$, and therefore $\psi_t = \psi - t\delta$ and we may rewrite (2) as

$$V_t \cdot (\psi - t\delta) = -\delta.$$

Consider for each $g \in \mathcal{E}$ and $\forall t \in [0,1]$ the initial value problem given by

$$V_t(0) = 0 \tag{3}$$
$$V_t \cdot (\psi - t\delta) = g. \tag{4}$$

Lemma 2.10. *There exists a vector field* V_t *that satisfies (4)* $\forall t \in [0,1], g \in \mathfrak{m}$. *If* $g \in \mathfrak{m}^2$, *then some* V_t *satisfies (3) as well.*

Proof. The space of solutions to (4) is linear, and we can write any element $g \in \mathfrak{m}$ in terms of the coordinates x^i. Therefore it is enough to consider solutions V_t^i of

$$V_t \cdot (\psi - t\delta) = x^i.$$

By Lemma 2.9, there exist $a_{ij} \in \mathcal{E}$ such that $x^i = \sum_j a_{ij}\partial_{x^j}\psi$. In matrix form, we have

$$x = A(x)\nabla_\psi \iff x = A(x)\nabla(\psi - t\delta) + tA(x)\nabla\delta. \tag{5}$$

Then by Lemma 2.9, we have that $\delta \in \mathfrak{m}^3 \implies \partial_{x^i}\delta \in \mathfrak{m}^2\ \forall i$. We may then write $\partial_{x^i}\delta$ in terms of the coordinates x^i;

$$\partial_{x^i}\delta = \sum_j b_{ij}x^j$$

for some $b_{ij} \in \mathfrak{m}$. Therefore, we have in matrix form

$$\nabla\delta = Bx, \quad B(0) = 0.$$

Combining this with (5), we have

$$(\mathbf{I} - tA(x)B(x))x = A(x)\nabla(\psi - t\delta).$$

Then $(\mathbf{I} - tA(x)B(x))$ is invertible for small enough x since $B(0) = 0$, and we denote this by

$$C_t(x) := (\mathbf{I} - tA(x)B(x))^{-1}.$$

Now $x = C_t(x)A(x)\nabla(\psi - t\delta)$, and therefore

$$x^i = \sum_j V_j^i(t,x)\partial_{x^j}(\psi - t\delta)$$

where we have written $V_j^i(t,x)$ for the $(i,j)^{th}$ entry of the matrix $C_t(x)A(x)$.

Thus, we have that $V_t^i = \sum_j V_j^i(t,x)\partial_{x^j}$ is a solution of

$$V_t \cdot (\psi - t\delta) = x^i.$$

If $g = \sum_i g_i x^i \in \mathfrak{m}$, then $\sum_i g_i V_t^i$ is a solution of (4), and furthermore if $g \in \mathfrak{m}^2$, we can choose $g_i \in \mathfrak{m}$ so that we also have a solution of (3). $\square$

Finally, since $\delta \in \mathfrak{m}^3 \subset \mathfrak{m}^2$, there exists a solution V_t of (2), which proves Theorem 2.7. $\square$

The Morse Lemma follows as a result of the following fact from linear algebra. [3]

Lemma 2.11. *Let V be a real vector space, and $b : V \times V \to \mathbb{R}$ a symmetric, nondegenerate bilinear map. Then there is a basis $(e_1, ...e_n)$ of V such that for some $v = \sum_{i=1}^{n} v^i e_i \in V$,*

$$b(v, v) = -\sum_{i=1}^{\lambda} |v^i|^2 + \sum_{i=\lambda+1}^{n} |v^i|^2.$$

Therefore, we may find suitable coordinates so that the Hessian takes the form as above.

□

Example 2.12 (The torus). A simple example, as discussed in [6] is the torus $\mathbb{T}^2$. The height function on $\mathbb{T}^2$ is a Morse function with four critical points, the global maximum and minimum and the two saddle points as shown in Figure 1. Let the critical points be labelled p_0, p_1, p_2, p_3 with increasing critical value.

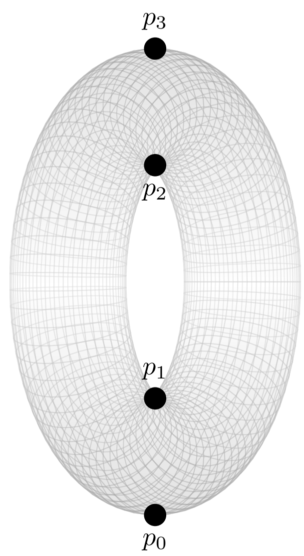


Figure 1: Critical points of the height function on a torus.

In order to calculate the indices,we will need to calculate the Hessian matrices in local coordinates.

We may write, at each critical point, the function f in the form $f = \pm x^2 \pm y^2$.
In particular, at the global maximum and minimum, the torus locally looks like a paraboloid. Similarly, we know the parameterisation of a saddle. Thus we have

$$f(x,y) = \begin{cases} x^2 + y^2 & at\ p_0 \\ x^2 - y^2 & at\ p_1, p_2 \\ -x^2 - y^2 & at\ p_3. \end{cases}$$

Therefore we get the hessian matrices

$$H_{p_0}(f) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad H_{p_1}(f) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad H_{p_2}(f) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad H_{p_3}(f) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Thus, the indices of the critical points are $0, 1, 1$ and 2 respectively. Note that these local representations are of the form described by the Morse Lemma, and so we could also read off the indices from the local parameterisation $f(x, y)$.

3 Homotopy Type

One of the most fundamental results coming directly from a Morse function f on a manifold M is the relationship between the homotopy type of M and the critical points of f. By slicing M along the level sets of f, properties of the Morse function can tell us how the associated sublevel sets change as we pass critical points. The following main results are described in detail by Milnor [6], whose notation we will also use.

For the rest of the chapter, we will let $f : M \to \mathbb{R}$ be a Morse function on a manifold M.

Theorem 3.1. *If for $a < b \in \mathbb{R},\ f^{-1}[a,b] = \{q \in M \mid a \leq f(q) \leq b\}$ is compact and contains no critical points of f, then the sublevel set $M^a := f^{-1}(-\infty, a]$ is diffeomorphic to $M^b := f^{-1}(-\infty, b]$. In fact M^a is a deformation retract of M^b, and the inclusion map from M^a into M^b is a homotopy equivalence.*

This says that the homotopy type of the sublevel set $f^{-1}(-\infty, y],\ y \in \mathbb{R}$, does not change as y is continuously varied without passing any critical values.

Proof. This theorem can be proved by pushing M^b back to M^a along trajectories orthogonal to the level sets of f.

We first need a Riemannian metric on M, so that we can define these trajectories in terms of a gradient vector field. Denote by

$$X(f) = \langle X, grad\ f\rangle$$

the vector field $grad\ f$. Then we can see that this vector field only has zeros at the critical points of f.

For a curve $c : \mathbb{R} \to M$, we have

$$\left\langle \frac{dc}{dt}, grad\ f \right\rangle = \frac{d(f \circ c)}{dt}.$$

Define the vector field

$$X_q = \begin{cases} \frac{(grad\ f)_q}{\langle grad\ f, grad\ f\rangle} & if\ q \in f^{-1}[a,b] \\ 0 & if\ q \in M \setminus N([a,b]) \end{cases}$$

for $N([a,b])$ a compact neighbourhood of $[a,b]$. Then by a result in Riemannian geometry [8], a smooth vector field of this form generates a unique 1-parameter group of diffeomorphisms of M. That is, a map $\phi : \mathbb{R} \times M \to M$ such that $\forall t \in \mathbb{R}$, $\phi_t(q) := \phi(t, q)$ is a diffeomorphism of M onto M satisfying

$$\phi_{s+t} = \phi_s \circ \phi_t \ \forall s, t \in \mathbb{R}.$$

We have that

$$X_q(f) = \lim_{h \to 0} \frac{f(\phi_h(q)) - f(q)}{h}.$$

Then, whenever $\phi_t(q) \in f^{-1}([a, b])$,

$$\frac{df(\phi_t(q))}{dt} = \left\langle \frac{d\phi_t(q)}{dt}, grad\ f \right\rangle = \langle X, grad\ f\rangle = \left\langle \frac{grad\ f}{\langle grad\ f, grad\ f\rangle}, grad\ f \right\rangle = 1.$$

So, $t \to f(\phi_t(q))$ is linear with derivative 1 whenever $\phi_t(q) \in f^{-1}([a, b])$.

Then $\phi_{b-a} : M \to M$ is a diffeomorphism that takes M^a to M^b, hence M^a and M^b are diffeomorphic. A retraction map from M^a to M^b follows from the 1-parameter family of maps $r_t : M^b \to M^b$ given by

$$r_t(q) = \begin{cases} q & if\ f(q) \le a \\ \phi_{t(a-f(q))}(q) & if\ a \le f(q) \le b. \end{cases}$$

We can see that r_0 is the identity, since either $r_0(q) = \phi_0(q)$ or $r_0(q) = q$. We know that $\phi_0(q) = q \ \ \forall q$, as a result of the property $\phi_{t+0} = \phi_t \circ \phi_0 = \phi_t \ \forall t \in \mathbb{R}$.

Also, to show that r_1 is a retraction from M^b to M^a, we need that $r_1(q) \in M^a \ \ \forall q \in M^b$. It is clear that r_1 is identity on M^a since if $q \in M^a$, $r_1(q) = q \in M^a$. Due to the linearity of $t \to f(\phi_t(q))$ and the fact it has gradient 1 for all $\phi_t(q) \in f^{-1}[a, b]$, we know that $f(\phi_{a-f(q)}(q))$ varies linearly with $a - f(q)$, and in particular,

$$a - b \le f(\phi_{a-f(q)}(q)) \le a \implies r_1(q) \subset f^{-1}[a - b, a] \subset M^a.$$

Therefore, M^a is a deformation retract of M^b. □

Theorem 3.2. *Suppose p is a critical point of f with index λ, and $f(p) = c$. If $f^{-1}[c - \epsilon, c + \epsilon]$ for some $\epsilon > 0$ is compact and contains no critical point other than p, then for ϵ small enough, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

Proof. Using the method in [6, 3]. We begin by using the properties of a Morse function in order to find a new function F that satisfies a number of useful properties. In particular, the new function we will define will coincide with the Morse function $f : M \to \mathbb{R}$ except in a small neighbourhood of each critical point p_i, where it will take smaller values than the original Morse function.

Then we will have that the sublevel sets $F^{-1}(-\infty, c-\epsilon]$ are an extension of the regions $M^{c-\epsilon} = f^{-1}(\infty, c-\epsilon]$, with an attached small region containing the critical point p_λ of index λ. Then it can be shown that $f^{-1}(\infty, c-\epsilon] \cup e^\lambda$ is a deformation retract of $F^{-1}(-\infty, c-\epsilon]$, which is itself a deformation retract of $f^{-1}(\infty, c+\epsilon]$.

Let p be a critical point of f. Then there exists local coordinates $(x^1, x^2, ..., x^n)$ in a neighbourhood U of p such that for some $\epsilon > 0$ we can write f in U as

$$f = c - \sum_{i=1}^{\lambda}(x^i)^2 + \sum_{i=\lambda+1}^{n}(x^i)^2,$$

and f satisfies the following properties:

- $x^i(p) = 0 \; \forall i \in \{1, ..., n\}$ since p a critical point,
- $f^{-1}([c-\epsilon, c+\epsilon])$ is compact, containing p as its only critical point,
- The image of U under the diffeomorphism $(x^1, ..., x^n) : U \to \mathbb{R}^n$ contains the closed ball $D^n(0, 2\epsilon) := \{(x^1, ..., x^n) | \sum_{i=1}^n (x^i)^2 \leq 2\epsilon\}$.

Then we will construct our new function F by first defining a function $\mu(s) : \mathbb{R} \to \mathbb{R}$ such that the following is satisfied.

- $\mu(0) > \epsilon$,
- $\mu(s) = 0 \;\; for \; r \geq 2\epsilon$,
- $-1 < \frac{d\mu(s)}{ds} \leq 0 \;\; \forall s.$

Also, to simplify expressions, define $x^-, x^+ \subset U$ and the maps $\nu_-, \nu_+ : U \to \mathbb{R}^{\geq 0}$ by

$$x^- := (x^1, ..., x^\lambda, 0, ..., 0), \qquad \nu_- := \sum_{i=1}^{\lambda}(x^i)^2,$$
$$x^+ := (0, ..., 0, x^{\lambda+1}, ..., x^n), \qquad \nu_+ := \sum_{i=\lambda+1}^{n}(x^i)^2$$

so that $f = c - \nu_- + \nu_+$. Then, we can define

$$F = \begin{cases} f - \mu\left(\nu_- + 2\nu_+\right) := c - \nu_- + \nu_+ - \mu(\nu_- + 2\nu_+) & in \; U \\ f := c - \nu_- + \nu_+ \quad in \; M \setminus U \end{cases}$$

Lemma 3.3. *F is a Morse function on M with the same critical points as f.*

Proof. F is a smooth function, and we have

$$dF = \frac{\partial F}{\partial \nu_-} d\nu_- + \frac{\partial F}{\partial \nu_+} \nu_+$$

and $\begin{cases} \frac{\partial F}{\partial \nu_-} = -1 - \mu' \\ \frac{\partial F}{\partial \nu_+} = 1 - 2\mu' \end{cases} \implies dF = -(1+\mu')d\nu_- + (1-2\mu')d\nu_+.$

In particular, we have $\frac{\partial F}{\partial \nu_-} < 0$ and $\frac{\partial F}{\partial \nu_+} \geq 1$, since $-1 < \mu'(s) \leq 0 \; \forall r$.
Thus, if $dF = 0$ at a point q, we must have $d\nu_-(q) = d\nu_+(q) = 0$. This can only happen at the origin. Therefore, the only critical point of F in U is the origin. □

Lemma 3.4. $F^{-1}(-\infty, c+\epsilon] = f^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon}$

Proof. $F = f$ in $M \setminus U$. In fact, $F = f$ outside the region where $\nu_- + 2\nu_+ \leq 2\epsilon$, since $\mu(s) = 0$ for $s \geq 2\epsilon$. Inside the region, we have

$$F = c - \nu_- + \nu_+ - \mu(\nu_- + 2\nu_+) \leq c - \nu_- + \nu_+ = f \leq c + \frac{1}{2}\nu_- + \nu_+ \leq c + \epsilon.$$

The first inequality comes from the fact that $\nu_-, \nu_+ \geq 0$ and $-1 < \mu'(s) \leq 0$, so for $0 \leq \nu_- + 2\nu_+ \leq 2\epsilon$, we have

$$\epsilon < \mu(\nu_- + 2\nu_+) \leq 0.$$

Thus,

$$F^{-1}(-\infty, c+\epsilon] \cap \{\nu_- + 2\nu_+ \leq 2\epsilon\} = M^{c+\epsilon} \cap \{\nu_- + 2\nu_+ \leq 2\epsilon\}$$

and elsewhere, f and F coincide. □

Lemma 3.5. $F^{-1}(-\infty, c-\epsilon]$ *is a deformation retract of* $M^{c+\epsilon}$.

Proof. Note that $F^{-1}[c-\epsilon, c+\epsilon] \subset f^{-1}[c-\epsilon, c+\epsilon]$. This is because Lemma 3 and the fact $F \leq f$ imply that if $F(q) \geq c - \epsilon$, then $f(q) \geq F(q) \geq c - \epsilon$.

Therefore, as a closed subset of a compact set, $F^{-1}[c-\epsilon, c+\epsilon]$ is compact. The only critical point it could contain would be a critical point in $f^{-1}[c-\epsilon, c+\epsilon]$ which must be p.
However,

$$F(p) = c - \mu(0) < c - \epsilon \notin F^{-1}[c-\epsilon, c+\epsilon],$$

which implies that $F^{-1}[c-\epsilon, c+\epsilon]$ contains no critical points.

By Theorem 3.1 and Lemma 3, $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $F^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon}$. □

Let the degree λ cell e^λ, where λ is the index of the critical point p, be defined by

$$e^\lambda = \{q \in M | \nu_-(q) \leq \epsilon, \; \nu_+(q) = 0\}.$$

Then e^λ is contained in the small region $F^{-1}(-\infty, c-\epsilon] \setminus M^{c-\epsilon} := H$. It is also true that $\forall q \in e^\lambda$,

$$F(q) \le F(p) < c - \epsilon, \quad and \;\; f(q) \ge c - \epsilon.$$

This is because

$$F(q) = c - \nu_-(q) - \mu(\nu_-(q)) \le F(p) = c - \mu(0) < c - \epsilon.$$

We have $F(q) \le F(p)$ since $\frac{\partial F}{\partial \nu_-} < 0$ and $0 = \nu_-(p) \le \nu_-(q)$.

The final thing to prove is that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H$.

This can be proved by constructing a deformation retraction

$$r_t(q) : M^{c-\epsilon} \cup H \to M^{c-\epsilon} \cup H$$

such that it is identity outside U. To define r_t in U, we split the region U in to three regions: $U_1 := \{\nu_- < \epsilon\}$, $U_2 := \{\epsilon \le \nu_- \le \nu_+ + \epsilon\}$ and $U_3 := \{\nu_+ + \epsilon \le \nu_-\}$.

- In U_1, take

  $$r_t : (x^1, ..., x^n) \mapsto (x^1, ..., x^\lambda, tx^{\lambda+1}, ..., tx^n) := x^- + tx^+$$

  $\implies r_1$ is the identity map, and

  $$r_0 : U_1 \to e^\lambda, \quad with \quad r_0(x^1, ..., x^n) = x^-.$$

  Since for x_- we have $\nu_+ = 0$, it is clear that $x^- \in e^\lambda$.
  Finally, $r_t(q)$ maps the region $M^{c-\epsilon} \cup H = F^{-1}(\infty, c-\epsilon]$ to itself since we have $\frac{\partial F}{\partial \nu_+} > 0$. We have that ν_+ increases with t, and hence F increases with t. Hence for $t \le 1$, we have for $q \in F^{-1}(\infty, c-\epsilon]$ that

  $$F(r_t(q)) \le F(r_1(q)) = F(q) \le c - \epsilon.$$

- In U_2, let

  $$r_t : (x^1, ..., x^n) \mapsto (x^1, ..., x^\lambda, \zeta_t x^{\lambda+1}, ..., \zeta_t x^n),$$

  where $\zeta_t \in [0,1]$ such that

  $$\zeta_t := t + (1-t)\left(\frac{\nu_- - \epsilon}{\nu_+}\right)^{1/2}.$$

  Then, r_1 is identity on U_2, and $r_0 : U_2 \to f^{-1}(c-\epsilon)$. This is because

  $$r_0(x^1, ..., x^n) = \left(x^1, ..., x^\lambda, \left(\frac{\nu_- - \epsilon}{\nu_+}\right)^{1/2} x^{\lambda+1}, ..., \left(\frac{\nu_- - \epsilon}{\nu_+}\right)^{1/2} x^n\right)$$

  $$\implies f(r_0(x^1, ..., x^n)) = c - \nu_- + \frac{\nu_- - \epsilon}{\nu_+}\nu_+ = c - \epsilon.$$

- In U_3, we may take $r_t(q) = q$.

  Then, we have

$$\nu_+ + \epsilon \leq \nu_- \iff \nu_- - \nu_+ \geq \epsilon \iff f := c - (\nu_- - \nu_+) \leq c - \epsilon.$$

  That is, the region coincides with $M^{c-\epsilon}$, so $r_t = id$ is a deformation retraction from $M^{c-\epsilon} \cup H$ to $M^{c-\epsilon} \cup e^\lambda$ when restricted to U_3.

Finally, r_t defined on U_2 corresponds to r_t defined on U_1 when $\nu_- = \epsilon$. Also, r_t as defined on U_3 is the same as r_t on U_2 when $\nu_- = \nu_+ + \epsilon$.
Hence, $r_t : M^{c-\epsilon} \cup H \to M^{c-\epsilon} \cup H$ is a deformation retraction, and $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H$.

Thus, we have shown that $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $M^{c+\epsilon}$, and in Lemma 3.5 that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H = F^{-1}(-\infty, c-\epsilon]$. Therefore, we have that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c+\epsilon}$, and in particular they have the same homotopy type. This proves Theorem 3.2. □

The following, final theorem regarding the homotopy type of M is of particular relevance since it gives us a cellular structure for the manifold M.

Theorem 3.6. *Suppose M^a is compact $\forall a \in \mathbb{R}$. Then M has the homotopy type of a CW-complex with a λ-cell for each critical point of index λ.*

The proof of this theorem follows from what we have proved above, along with the following lemmas. [6]

Lemma 3.7. *Let X be a topological space. Suppose that $f : S^{\lambda-1} \to X, g : S^{\lambda-1} \to X$ are homotopic maps. Then the identity map on X extends to the homotopy equivalence*

$$h : X \underset{f}{\cup} e^\lambda \to X \underset{g}{\cup} e^\lambda.$$

The space

$$X \underset{f}{\cup} e^\lambda := (X \amalg e^\lambda)/\sim$$

is given by the disjoint union of the λ-cell e^λ and X, with the gluing relationship characterised by the equivalence relation given by $p \sim f(p)$, for $p \in S^{\lambda-1}$.

Lemma 3.8. *Let $f : S^{\lambda-1} \to X$ be an attaching map. Then for two homotopy equivalent topological spaces X and Y, the homotopy equivalence $h : X \to Y$ extends to the homotopy equivalence*

$$H : X \underset{f}{\cup} e^\lambda \to Y \underset{hf}{\cup} e^\lambda.$$

Proof of Theorem 3.6. Since M^a is compact $\forall a \in \mathbb{R}$, we have for $c_1 < c_2 < ...$, the critical values of the Morse function f, that the sequence (c_i) has no cluster points. Suppose that a is not a critical point, and that M^a is homotopy equivalent to some CW-complex. Then $c_{i-1} < a < c_i$ for some $i \in \mathbb{N}$. Denote $c := c_i$. We will also assume that there is a homotopy equivalence $h' : M^a \to X$ for some CW-complex X.
By combining the previous results, we see that there exists an $\epsilon > 0$ such that the homotopy type of $M^{c+\epsilon}$ is the same as that of

$$M^{c-\epsilon} \underset{f_1}{\cup} e^{\lambda_1} \underset{f_2}{\cup} e^{\lambda_2} \cup ... \underset{f_m}{\cup} e^{\lambda_m}$$

for some m.
Here, the λ_k are the indices of the m critical points in $f^{-1}(c)$, and the f_k are attaching maps.

There is a homotopy equivalence $h : M^{c-\epsilon} \to M^a$, since there are no critical points in the region $f^{-1}[a, c-\epsilon]$.

Then we have that $h' \circ h \circ f_k$ is homotopic to a map $\psi_k, \forall k$ such that

$$\psi_k : S^{\lambda_k - 1} \to (\lambda_k - 1)\text{–}\mathit{skeleton\ of\ } X.$$

$X \underset{\psi_1}{\cup} e^{\lambda_1} \cup ... \underset{\psi_m}{\cup} e^{\lambda_m}$ is a CW-complex. By Lemmas 3.7 and 3.8, it has the same homotopy type as $M^{c+\epsilon}$ since we know that X is homotopy equivalent to M^a which is homotopy equivalent to $M^{c-\epsilon}$, and since we have homotopic maps.

By the same method, we see that this holds for all a, that is, M^a has the homotopy type of a CW-complex $\forall a \in \mathbb{R}$.

If M is not compact, but has finitely many critical points, then they all lie in a compact set M^a for some a. It can be shown similarly to above that M^a is a deformation retract of M, since there are no critical points in $f^{-1}[a, \infty)$.

If there are infinitely many critical points of the function $f : M \to \mathbb{R}$, then the sequence (c_i) is infinite, and it can be shown that M is again homotopy equivalent to a CW-complex by considering the limit of the homotopy equivalences $h_i : M^{a_i} \to X_i, \;\; i \in \mathbb{N}$.

In any case, we have constructed a homotopy equivalence from M to a CW-complex with a cell of dimension λ for each critical point of index λ. $\square$

3.1 Examples

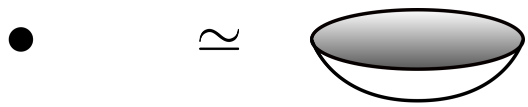
To demonstrate the above results regarding homotopy type, let us see what it tells us about a couple of simple examples, namely, the torus $\mathbb{T}^2$, and the complex projective space $\mathbb{CP}^n$.

Example 3.9 (Homotopy type of $\mathbb{T}^2$). We already determined that the height function on $\mathbb{T}^2$ is Morse, and has 4 critical critical points of indices 0, 1, 1 and 2.

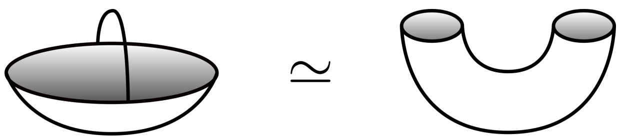
Therefore, by the theorems above, this tells us that the torus is homotopy equivalent to a CW-complex comprised of one 0-cell, two 1-cells and a 2-cell. That is,

$$\mathbb{T}^2 \simeq e^0 \cup e^1 \cup e^1 \cup e^2.$$

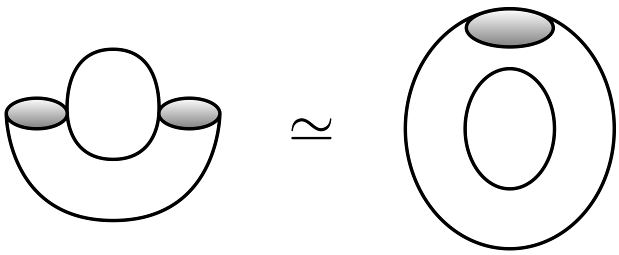
This can easily be seen by the following construction.



We begin with the empty set. As we pass the first critical point of index 0, we add a 0-cell, that is a point. This is homotopy equivalent to a disk.



Then as we pass the second critical point, with index 1, we add a 1-cell or handle as shown below.



The third critical point also has index 1, and another handle is added.

Then the final critical point has index 2, and attaching a 2-cell to the above diagram gives us the torus.

Example 3.10 (Homotopy type of $\mathbb{CP}^n$). Consider the space

$$\mathbb{CP}^n := \mathbb{C}^{n+1}/\mathbb{C}^{\times}$$

where we have the equivalence relation

$$(z_0, ..., z_n) \sim (\lambda z_0, \lambda z_1, ..., \lambda z_n) = \lambda(z_0, ..., z_n)$$

for any $\lambda \in \mathbb{C}^{\times}$. As is explained in [6], we have that

$$f(z_0, ..., z_n) = \sum_{i=0}^{n} c_i |z_i|^2$$

where $c_i \in \mathbb{R}$ are constants, is a Morse function on $\mathbb{CP}^n$. In the local coordinates $(z_0, ...z_n)$, f can be written in terms of z_0 for some only by the implicit function theorem as long as $z_0 \neq 0$.

Write

$$|z_0| \frac{z_j}{z-0} = x_j + iy_j \implies |z_0|^2 = 1 - (x_j^2 + y_j^2) \ \forall j \neq 0,$$

and f can be rewritten as

$$f = c_0 + \sum_{j=1} (c_j - c_0)(x_j^2 + y_j^2)$$

Define $(c_j - c_0) =: r_j$. Hence we have

$$df = 2[r_1 x_1, ..., r_n x_n, r_1 y_1, ..., r_n y_n]$$

So for a critical point, we have $z_j = 0 \ \forall j \neq k$. Since $\lambda \mathbf{z} \sim \mathbf{z}$, we have one critical point: $c_k = (0, ..., 1, ..., 0) \in \mathbb{CP}^n$ where the 1 in is the z_k^{th} coordinate. This is true for all $k \in \{0, ..., n\}$ and so in total there are $n + 1$ critical points p_i of this form.

The index can be found from the Hessian matrix

$$H_{p_0}(f) = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array}\right], \ where\ M = \begin{bmatrix} 2r_1 & 0 & 0 & ... & 0 \\ 0 & 2r_2 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 2r_n \end{bmatrix}.$$

Since the above method of eliminating z_0 can be repeated for each other coordinate z_j, we may assume without loss of generality that we have an ordering $c_0 \leq c_1 \leq ... \leq c_n$. Then the dimension of $H_{p_0}(f)$ that is negative definite is 0. Similarly, the dimension of $H_{p_n}(f)$ that is negative definite is $2n$.

For each $H_{p_k}(f)$, by definition of the coefficients $r_j := c_j - c_k$, we see that the dimensions of the negative-definite subspaces of the Hessians $H_{p_k}(f)$ will run through all elements of the set $\{0, 2, 4, .., 2n\}$. These integers are by definition the indices of the critical points p_k.

Therefore, we have the result that

$$\mathbb{CP}^n \sim e^0 \cup e^2 \cup e^4 \cup ... \cup e^{2n}.$$

4 The Morse Inequalities

The Morse inequalities are a fundamental result in Morse theory. They describe a relationship between the *Betti numbers* of a manifold M, and the critical points of the Morse function f on M. To understand this relationship, let us first define the Betti numbers of a space.

Definition 4.1 (Betti numbers). Let M be a smooth compact manifold. Then there are integers $b_k(M)$ called *Betti numbers* defined by

$$b_k(M) := rk(H_k(M))$$

where $H_k(M)$ is the k^{th} singular homology group of M. [11]

Informally, the Betti numbers relate to the number of "holes" in M. For example, $b_0(M)$ gives the number of connected components of M, and $b_1(M)$ gives the number of 1-dimensional holes in M.

Before we state the Morse inequalities, there are some necessary definitions that will help us to reduce the statement to a simpler one in the proof.

Definition 4.2 (Morse polynomial). For a Morse function $f : M \to \mathbb{R}$ on an n-dimensional manifold M, define the *Morse polynomial*

$$M_t(f) := \sum_{k=0}^{n} c_k t^k.$$

[5]

Definition 4.3 (Poincaré polynomial). The *Poincaré polynomial* of a manifold M of dimension n is defined as

$$P_t(M) = \sum_{k=0}^{n} b_k(M) t^k.$$

[5]

Theorem 4.4 (Weak Morse Inequalities). *Let M be a compact manifold and f a Morse function on M. Let c_k denote the number of critical points of f with index k, and let $b_k(M)$ denote the k^{th} Betti number of M. Then*

$$b_k(M) \leq c_k.$$

[6, 5]

Proof. The weak Morse inequalities follow from the definitions of c_k and b_k, our knowledge of the homotopy type of M, and the use of cellular homology. Proved as in [5].

We know that $\forall k \in 1, ..., n$,

$$H_k^{CW}(M^k, M^{k-1}) \cong \mathbb{Z}^{c_k},$$

since c_k is the number of cells of dimension k. We also have

$$b_k = dim(H_k(M)) = dim(H_k^{CW}(M))$$

since it is true that the cellular homology is homeomorphic agrees with the singular homology of M.

$H_k^{CW}(M) := ker(d_k)/im(d_{k+1})$ where the map d_r is defined by the composition

$$\begin{aligned} & H_m(M^r, M^{r-1}) \to H_{r-1}(M^{r-1}) \to H_{r-1}(M^{r-1}, M^{r-2}) \\ \iff \; & \mathbb{Z}^{c_r} \to H_{r-1}(M^{r-1}) \to \mathbb{Z}^{c_{r-1}} \end{aligned}$$

Thus, we have

$$b_k := dim\ ker(d_k)/im(d_{k+1}) \leq dim\ ker(d_k) \leq dim(\mathbb{Z}^{c_k}) = c_k \iff b_k \leq c_k \ \ \forall k.$$

□

Theorem 4.5 (Polynomial Weak Morse Inequalities). *In terms of the Morse polynomial and Poincaré polynomial we have*

$$\begin{aligned} \sum_{k=0}^{n} b_k(M) &\leq \sum_{k=0}^{n} c_k \\ \iff P_1(M) &\leq M_1(f). \end{aligned}$$

[6]

Theorem 4.6 (Morse Inequalities). *[6] [5]*

$$\sum_{k=0}^{n} (-1)^k b_k(M) = \sum_{k=0}^{n} (-1)^k c_k \tag{6}$$

$$\sum_{k=0}^{s} (-1)^{k+s} b_k(M) \leq \sum_{k=0}^{s} (-1)^{k+s} c_k \quad \forall s \in \{0, ..., n\} \tag{7}$$

We will follow the proof in [5], using an equivalent relation of polynomials.

Proof of the Morse inequalities. To prove the Morse inequalities, we reduce the statement to the following statement regarding the Morse and Poincaré polynomials.

Theorem 4.7 (Polynomial Morse Inequalities).

$$M_t(f) = P_t(M) + (1+t)R(t) \tag{8}$$

where $R(t)$ is a polynomial with coefficients $r_i \geq 0 \ \ \forall i \in \{0, ..., n-1\}$.

Proof of equivalence. We can show that the two statements are equivalent by using the definitions of the polynomials $M_t(f)$ and $P_t(M)$. (6) gives us

$$\begin{aligned}
P_{-1}(M) = \sum_{k=0}^{n} (-1)^k b_k(M) = \sum_{k=0}^{n} (-1)^k c_k = M_{-1}(f) \\
\Rightarrow (1-t) \mid (M_t(f) - P_t(M)) \\
\Rightarrow M_t(f) = P_t(M) + (1+t)R(t).
\end{aligned}$$

The coefficients r_i of $R(t)$ are integers since the coefficients of $M_t(f)$ and $P_t(M)$ are integers. The weak Morse inequalities will require that the coefficients are non-negative. By considering the equation (8) for different values of t we get the following.

$$t = 0: \quad c_0 = b_0 + r_0. \tag{9}$$

Now, if we expand (8) and consider just the linear terms at $t = 1$ we get

$$\begin{aligned}
c_0 + c_1 t = b_0 + b_1 t + r_0 + r_0 t + r_1 t \\
\textit{at } t = 1: \iff c_1 + c_0 = b_1 + r_1 + 2r_0 + b_0.
\end{aligned}$$

Using the relationship (9) to remove r_0 gives us

$$c_1 - c_0 = b_1 + r_1 - b_0.$$

We will now use induction to find the result for general $m \in \{0, ..., n-1\}$. We assume the result for all $m \leq k$ as follows

$$r_m = \sum_{i=0}^{m-i} (-1)^i c_i - \sum_{i=0}^{m-i} (-1)^i b_i.$$

Then, as before, we consider the terms of orders k and $k+1$ in (8) and set $t = 1$.

$$c_k + c_{k+1} = b_k + b_{k+1} + r_{k-1} + 2r_k + r_{k+1}.$$

Now, we substitute in the expressions for r_{k-1} and r_k from the inductive hypothesis and cancel terms.

$$\begin{aligned}
c_k + c_{k+1} = b_k + b_{k+1} + \sum_{i=0}^{k-1} (-1)^{(k-1)-i} (c_i - b_i) + 2 \sum_{i=0}^{k} (-1)^{k-i} (c_i - b_i) + r_{k+1} \\
\iff c_k + c_{k+1} = b_k + b_{k+1} + 2(c_k - b_k) - \sum_{i=1}^{k-1} (-1)^{k-i} (c_i - b_i) + r_{k+1} \\
\iff \sum_{i=0}^{k+1} (-1)^{(k+1)-i} c_i - \sum_{i=0}^{k+1} (-1)^{(k+1)-i} b_i = r_{k+1}.
\end{aligned}$$

Hence, we have proved the statement $\forall m \in \{0, ..., n-1\}$. From this, it is clear that the r_i are non-negative $\forall i \in \{0, ..., n-1\}$ by the weak morse inequalities $b_k(M) \leq c_k$. $\square$

Thus, it is left to prove the equivalent polynomial inequalities.

Proof of the polynomial Morse inequalities. Let $C_k(M;\mathbb{F})$ be the finitely generated chain complex generated by index k critical points of f, with $C_k = 0 \;\; \forall k > n$, and let ∂_k be the associated boundary map. Then the following sequence is a short exact sequence by definition.

$$0 \to ker\ \partial_k \to C_k(M;\mathbb{F}) \xrightarrow{\partial_k} im\ \partial_k \to 0$$

Define

$$\begin{aligned} v_k &:= rank\ ker\ \partial_k \\ w_k &:= rank\ im\ \partial_k. \end{aligned}$$

Then, $c_k = v_k + w_k \quad \forall k \in \{0, ..., n\}$, since $im\ \partial_k \cong C_k(M;\mathbb{F})/ker\ \partial_k$ by exactness, and $c_k = rank\ C_k(M;\mathbb{F})$ by definition.

Similarly, the short exact sequence

$$0 \to im\ \partial_{k+1} \to ker\ \partial_k \to H_k(M;\mathbb{F}) \to 0$$

implies that $b_k = v_k - w_{k+1} \quad \forall k \in \{0, ..., n\}$. This is because b_k is by definition the rank of $H_k(M;\mathbb{F})$, and by exactness, $H_k(M;\mathbb{F}) \cong ker\ \partial_k / im\ \partial_{k+1}$.

We can now substitute the above expressions for c_k and b_k into the left hand side of (8).

$$\begin{aligned} M_t(f) - P_t(M) &= \sum_{k=0}^{n} c_k t^k - \sum_{k=0}^{n} b_k t^k \\ &= \sum_{k=0}^{n} (c_k - b_k) t^k \\ &= \sum_{k=0}^{n} (v_k + w_k - v_k + w_{k+1}) t^k \\ &= \sum_{k=0}^{n} (w_k + w_{k+1}) t^k \\ &= \sum_{k=0}^{n} (c_k - v_k + c_{k+1} - v_{k+1}) t^k \\ &= t \sum_{k=0}^{n} (c_k - v_k) t^{k-1} + \sum_{k=1}^{n} (c_{k+1} - v_{k+1}) t^{k-1}. \end{aligned}$$

Then since $c_0 = v_0$, and relabelling the sum index in the second term, the following holds.

$$(t+1)\sum_{k=1}^{n}(c_k - v_k)t^{k-1}$$

which gives the desired result with $R(t) = \sum_{k=0}^{n-1}(c_{k+1} - v_{k+1})t^k$.

The coefficients $r_k = c_{k+1} - v_{k+1}$ are non-negative since $ker\ \partial_{k+1}$ is a subgroup of $C_{k+1}(M;\mathbb{F})$ which has rank c_{k+1}. $\square$

An alternative proof is presented in [3, 6], which uses the cellular structure determined by the homotopy results of the previous chapter, and applies properties of relative singular homology to the sublevel sets of M.

5 Morse Homology

One can determine the homology groups of a manifold M using only knowledge of a Morse function on the space.

The most common method of determining Morse homology is from the *Morse-Smale-Witten chain complex.* This is built from the critical points of a Morse function f, and the gradient flow between them. First, let us define some terms relating to the gradient flow.

5.1 Gradient Flow

To define a gradient flow, we choose a metric g on M.

Definition 5.1 (Gradient flow). Let $f : M \to \mathbb{R}$ be a Morse function. For $q \in M$, a gradient flow line through q is the maximal curve $\gamma(t)$ such that $\gamma(t_0) = q$ for some $t_0 \in \mathbb{R}$ and

$$\dot{\gamma}(t) = -\nabla f(\gamma(t)).$$

A flow line $\gamma(t)$ from p to q satisfies $\lim_{t\to-\infty} \gamma(t) = p$, and $\lim_{t\to+\infty} \gamma(t) = q$. [14]

In other words, the gradient flow of f describes the flow of a vector field V, where V is the negative gradient of the function f.

Definition 5.2 (Stable and unstable manifolds). Let ψ_s be the flow of the vector field V as defined above. Then $\psi_0 = id$ and $\dfrac{\mathrm{d}\psi_s}{\mathrm{d}s} = V$. Define

$$W^s(p) = \{q \in M \mid \lim_{s\to+\infty} \psi_s(q) = p\},$$
$$W^u(p) = \{q \in M \mid \lim_{s\to-\infty} \psi_s(q) = p\}$$

as the *stable* and *unstable manifolds* respectively. [15]

In simple terms, the stable manifold is the set of points that will be sent by the flow to the point p, and the unstable manifold is the set of points that are sent along the flow away from the point p.

Theorem 5.3 (Stable-unstable manifold theorem). *Let the stable and unstable manifolds be defined as above, and suppose $dim(M) = n$. Then for a critical point p with index λ_p of a Morse function $f : M \to \mathbb{R}$, we have*

$$T_pM = T_p^sM \oplus T_p^uM,$$

where $T_p^s M$, $T_p^u M$ are the tangent spaces of the stable and unstable manifolds of p. In other words, the tangent space to M at p splits into the subspaces on which the Hessian is positive and negative definite, which are $T_p^s M$ and $T_p^u M$ respectively.

Also, we have

$$dim(W^s(p)) = m - \lambda_p,$$
$$dim(W^u(p)) = \lambda_p,$$

and both $W^s(p)$ and $W^u(p)$ are smoothly embedded open disks given by the following embeddings.

$$E^s : T_p^s M \to W^s(p) \subset M$$
$$E^u : T_p^u M \to W^u(p) \subset M.$$

Furthermore, the above embeddings are homeomorphisms. [5]

A desirable property of the manifolds described above that is required for Morse homology to be defined is that they intersect *transversally.*

Definition 5.4 (Transverse intersection). Subspaces X, Y of a space M intersect *transversally* if

$$T_p(X) + T_p(Y) = T_p M \quad \forall p \in X \cap Y.$$

where $T_p(X) + T_p(Y) := \{u + v | u \in T_p X, v \in T_p Y\}$. We write $X \pitchfork Y$. [5]

5.2 Morse Homology Theorem

Definition 5.5 (Morse-Smale function). A Morse function $f : M \to \mathbb{R}$ is called *Morse-Smale* if for any pair of critical points p, q, the stable and unstable manifolds intersect transversally, i.e. $W_s(p) \pitchfork W_u(q)$. This is sometimes referred to as the *Morse-Smale transversality condition.* [5]

Definition 5.6. Let M be a smooth oriented manifold, and f a Morse-Smale function on M. Then to each flow γ from p to q we assign either the sign $+1$ or -1, using the orientation of M as follows.

For a point $x \in \gamma$, we have

$$-(\nabla(f))(\gamma(t)) = \frac{d}{dt}\gamma(t).$$

Given orientations of $W^u(q)$ and $W^s(p)$ at x, $-(\nabla(f))(x)$ may be completed to a positive basis $(-(\nabla(f))(x), \tilde{B}_x^u)$ of $T_x W^u(q)$. Then for any positive basis (B_x^s) of $T_x W^s(p)$, we have that $(\tilde{B}_x^u, B_x^s)$ is a basis of T_xM. The sign $+1$ is assigned to γ if the basis gives a positive orientation for T_xM, and the sign -1 otherwise.

Then define the integer $n(p,q) := \sum_\gamma sign(\gamma)$, the sum over all flow lines γ from p to q of these assigned signs. [16]

The Morse homology is obtained from the following *Morse chain complex*. This is also often referred to as the *Morse-Smale-Witten* or *Smale-Witten* complex.

Definition 5.7 (Morse-Smale-Witten chain complex). Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a smooth orientable manifold M. Then the Morse-Smale-Witten chain complex is defined as follows:

- The chain group $C_k(f)$ is the free abelian group generated by the critical points of f with index k.
- The homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ given by

$$\partial_k(q) := \sum_{p \in Cr_{k-1}(f)} n(p,q)p$$

  is the Morse-Smale-Witten boundary map. [5]

Theorem 5.8 (Morse homology theorem). *For a manifold M and Morse function f satisfying the properties as above, the homology of the Morse-Smale-Witten chain complex $(C_*(f), \partial_*)$ is isomorphic to the singular homology $H_*(M, \mathbb{Z})$. [5]*

Let us see how this works by considering the simple example of the torus $\mathbb{T}^2$.

Example 5.9 (The homology of $\mathbb{T}^2$). Consider the height function on the Torus as was mentioned in Chapter 2. As was calculated earlier, we have four critical points p, q, r, s of indices $0, 1, 1$, and 2 respectively.

In general, the height function we have defined on the torus is not Morse-Smale. This is because there are flow lines beginning at the critical point r of index 1 and ending at the other critical point q of index 1. The transversality condition implies that $W^u(r) \cap W^s(q)$ is a smooth submanifold with

$$dim(W^u(r) \cap W^s(q)) = \lambda_r + (m - \lambda_q) - m = \lambda_r - \lambda_q$$

where λ_r, λ_q are the indices of r and q respectively, and in this case $m = dim(\mathbb{T}^2) = 2$. However, since $W^u(r) \cap W^s(q) \neq \emptyset$, we must have

$$dim(W^u(r) \cap W^s(q)) = \lambda_r - \lambda_q \geq 1.$$

Therefore, we have a contradiction for the height function on the torus, however, by tilting the vertical axis of the torus slightly we can fix this problem. Everything else is then the same, as this rotation has no effect on the nature of the critical points.

The critical points and their indices give us three chain groups

$$C_0 = \mathbb{Z},\ C_1 = \mathbb{Z} \oplus \mathbb{Z},\ C_2 = \mathbb{Z}.$$

The Morse-Smale-Witten chain complex is therefore as follows:

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

To understand the maps ∂_k, we need to consider the flow lines between critical points. First we orient $\mathbb{T}^2$.

We have from the definition, and the fact that the chain groups C_i are generated by the critical points, that

$$\begin{aligned} \partial_2(s) &= (n(s,r)r, n(s,q)q) \\ \partial_1(q,r) &= n(q,p)p + n(r,p)p. \end{aligned}$$

However, we have that $n(s,q) = n(s,r) = n(q,p) = n(r,p) = 0$. For example, consider $n(s,r)$. This is the sum of the signs of the gradient flow lines from s to r. There are two such flow lines, in particular the flow around either side of the meridian containing s and r. Since the Torus is oriented, each of these will have opposite signs and so the sum is zero.
This is the case for the other pairs of critical points too, and therefore we get that both ∂_1 and ∂_2 are zero maps. This can be seen in Figure 2.

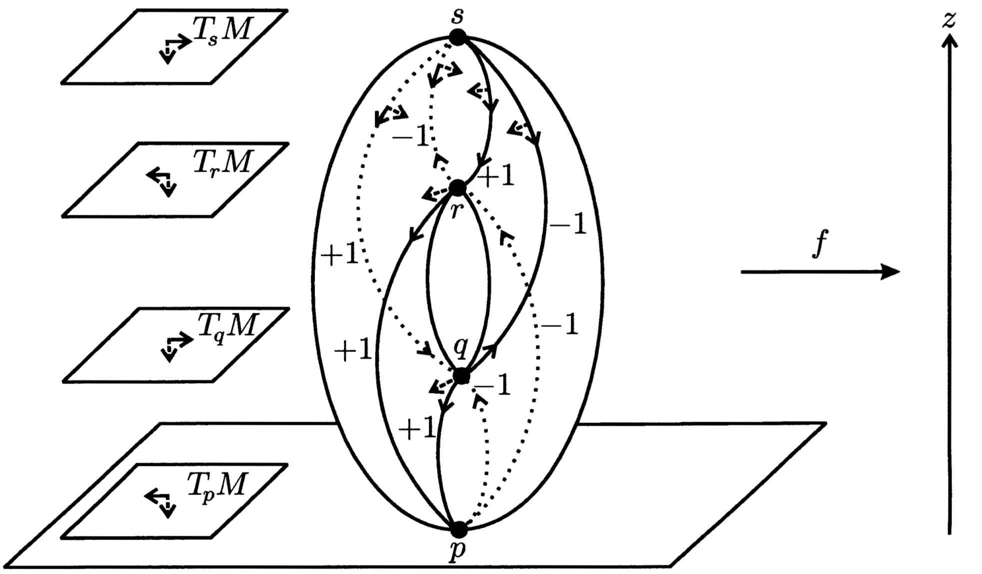


Figure 2: Flow lines between critical points on the torus. [5]

Hence, the homology groups are simply given by

$$
\begin{aligned}
H_0(\mathbb{T}^2) &\cong ker(\mathbb{Z} \to 0)/im(\mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}/0 = \mathbb{Z} \\
H_1(\mathbb{T}^2) &\cong ker(\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z})/im(\mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z})/0 = \mathbb{Z} \oplus \mathbb{Z} \\
H_2(\mathbb{T}^2) &\cong ker(\mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z})/im(0 \to \mathbb{Z}) = \mathbb{Z}/0 = \mathbb{Z}.
\end{aligned}
$$

[5]

6 Morse-Bott Functions

As has been discussed, many important results come from the study of Morse functions. However it may not always be easy to find a function on a given Manifold that satisfies all the conditions to be a Morse function.

By relaxing the condition on isolated critical points, and instead allowing functions with critical sets comprised of a collection of *critical sub-manifolds*, we can define a type of function called a *Morse-Bott function.* In many cases it can be preferable to work with Morse-Bott functions over Morse functions, and these can also provide some very useful information about the topology of the space.

6.1 Definitions

Definition 6.1 (Morse-Bott Function). A function $f : M \to \mathbb{R}$ is called a *Morse-Bott function* if for every critical value y, the set $f^{-1}(y)$ is a *nondegenerate critical submanifold* defined as follows.
A *nondegenerate critical submanifold* $C \subset M$ is a smooth manifold satisfying the following:

- every point in C is a critical point of the function f,
- C is a compact, connected manifold,
- $T_pC = kerH_{f,p} \;\; \forall p \in C$. That is,

$$H_{f,p}(X,Y) = 0 \;\; \forall Y \in T_pC \iff X \in T_pC \in T_pM.$$

As presented in [3].

Analogously to the Morse lemma, there is a result that says a Morse-Bott function can also be written as a quadratic form in some local coordinates. First, we will discuss some important definitions.

Definition 6.2. Let M be a smooth manifold.

- We define a *fibre bundle* E over M, where E is a smooth manifold, if there is a smooth map $\pi : E \to M$. Then $\forall q \in M$, we call $E_q \equiv \pi^{-1}(q)$ the *fibre* of E over q.
- A *vector bundle* is a special case of a fibre bundle. A real *vector bundle* of rank r over M is a fibre bundle where every fibre E_q is a real r-dimensional vector space. The vector spaces E_q vary smoothly with q.
- A tangent bundle TM is the disjoint union of tangent spaces to each point in M.

- Let $S \subset M$ be a smooth submanifold of M. Then the *normal bundle* N is the quotient of tangent bundles $(TM|_S)/TS$.
- A *disk bundle* $D(E)$ is a fiber bundle in which the fibres are disks.
- The *Thom space* $T(E)$ of a fibre bundle $\pi : E \to M$ is defined as the quotient

  $$T(E) := D(E)/S(E)$$

  where $D(E)$ and $S(E)$ are the disk bundle and sphere bundle of $\pi : E \to M$ respectively.

[17]

Definition 6.3 (Zero section). Let $\pi : E \to X$ be a vector bundle, and suppose X has dimension n.

- A map $s : X \to E$ given by

  $$s(x) = (x, f(x))$$

  where $f : X \to \mathbb{R}^n$ is an arbitrary map, is called a *section* of the vector bundle $\pi : E \to X$.
- The *zero section* is the section $X \to E$ sending every point in X to the zero vector over it in E.

[18]

Definition 6.4 (Tubular neighbourhood). Let M be a smooth manifold, and $A \subset M$ a submanifold. Then a *Tubular neighbourhood* of A is given by a vector bundle $\pi : E \to A$ and the embedding $\phi : E \to M$ that extends the diffeomorphism of the zero section Z onto A induced by π. We have $\phi(x, 0) = x$ for $(x, 0) \in Z$. [19]

Theorem 6.5 (Tubular neighbourhood theorem). *Let M be a smooth manifold, and $A \subset M$ a compact submanifold of X. Then A has a tubular neighbourhood in M, and moreover, any two tubular neighbourhoods $\phi : E \to M, \phi' : E' \to M$ of A in M are equivalent.*

That is, there exists an isotopy $H_t : E \to M$ such that $\phi = H_0$ and $\phi' = \psi H_1$, with $\psi : E \to E'$ a diffeomorphism that sends the vector space fibre E_x over x to E'_x by a linear isomorphism for each $x \in A$. [19, 20]

Lemma 6.6 (Morse-Bott Lemma). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function and suppose S is a connected component of the set of critical points $C_r(f)$ of f. Then for any point $p \in S$, there exists a local chart around p and a splitting of the normal bundle of S*

$$\nu_*(S) = \nu_*^+(S) \oplus \nu_*^-(S)$$

identifying x with (x, v, w), with $v \in \nu_*^+(S), w \in \nu_*^-(S)$, such that

$$f(x) = f(x, u, v) = f(S) + |v|^2 - |w|^2$$

within this chart. [5]

It follows from the Morse-Bott lemma that throughout each critical submanifold, the index remains constant. That is, $\forall$ critical submanifolds $C_i \subset Cr(f)$, if the index is λ for some point in C_i, then $index(p) = \lambda \ \ \forall p \in C_i$.

Example 6.7 (The torus). As we saw earlier, the height function on a Torus is a Morse function. If we rotate $\mathbb{T}^2$ as shown in Figure 3, then the height function is now a Morse-Bott function. It has two critical submanifolds, the global maximum and minimum are both copies of S^1.

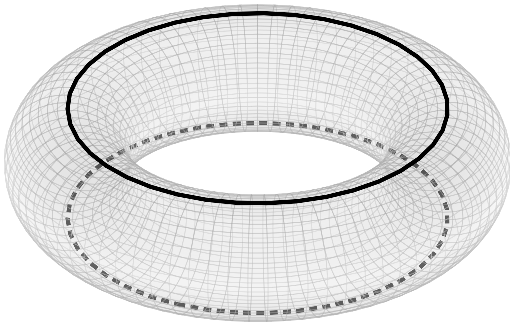


Figure 3: Critical submanifolds of the height function on a torus.

As with Morse functions, Morse-Bott functions have useful properties relating to the homotopy type of a given manifold. We will first define some useful terminology before stating the associated homotopy result.

6.2 Homotopy properties of Morse-Bott functions

Definition 6.8. Let X be a compact CW-complex, $x \in X$, and let $\mathbb{F}$ be a field.

- E is called $\mathbb{F}$-*orientable* if there is some cohomology class $\sigma = H^r(D(E), \partial D(E); \mathbb{F})$ such that the restriction of σ to each fiber $(D(E)_x, \partial D(E)_x)$ is a generator of the relative cohomology group $H^r(D(E)_x, \partial D(E)_x; \mathbb{F})$. In this case, σ is called the *Thom class* of E associated to some orientation.

- If E is the normal bundle of a critical submanifold C of a Morse Bott function, then define $E_-(C)$, the subspace of E spanned by the eigenvectors of the Hessian with negative eigenvalues. Note $E = E_+ \oplus E_-$, where E_+ is spanned by the eigenvectors with positive eigenvalues.
- A Morse-Bott function $f : M \to \mathbb{R}$ is $\mathbb{F}$-*orientable* if the bundle $E_-(C_i)$ is $\mathbb{F}$-orientable for each critical submanifold C_i.

[3]

For example, any Morse-Bott function $f : M \to \mathbb{R}$ is $\mathbb{Z}/2$-orientable since all vector bundles are $\mathbb{Z}/2$-orientable.

Theorem 6.9. *For a Morse-Bott function $f : M \to \mathbb{R}$, suppose c is a critical value such that $Cr_f \cap f^{-1}(c)$ is a collection of finitely many critical submanifolds $C_1, ..., C_k$. Let $D^-(C_i)$ be the closed unit disk bundle of $E_-(C_i)$. Then for some $\epsilon > 0$, $M^{c+\epsilon} := \{f \leq c+\epsilon\}$ is homotopy equivalent to the space $M^{c-\epsilon}$ with disk bundles $D^-(C_i)$ attached along their boundaries $\partial D^-(C_i)$. Therefore, the following is an isomorphism $\forall\ \mathbb{F}$. [3]*

$$H_*(M^{c+\epsilon}, M^{c-\epsilon}; \mathbb{F}) \cong \bigoplus_{i=1}^{k} H_*(D^-(C_i), \partial D^-(C_i); \mathbb{F}). \tag{10}$$

The above theorem gives an analogous result for a Morse-Bott function to the relationship between critical points of a Morse function and the CW structure of the manifold M. In the Morse-Bott case, the result is that M has the homotopy type of a collection of disk bundles. [21]

In other words, we have

$$M \sim D^-_{\lambda_1}(C_1) \underset{f_1}{\cup} D^-_{\lambda_2}(C_2) \underset{f_2}{\cup} ...D^-_{\lambda_{k-1}}(C_{k-1}) \underset{f_k}{\cup} D^-_{\lambda_k}(C_k)$$

where C_i are the critical submanifolds of f, and f_i map the boundary of each new disk bundle into the existing complex.

Example 6.10 (Structure of $\mathbb{T}^2$ from a Morse-Bott function). The height function on the torus gave us two critical submanifolds, both copies of S^1. The above theorem says that each of these contributes a cell bundle. These are $D^0(S^1)$ and $D^1(S^1)$, since the submanifolds have indices 0 and 1. Therefore we have

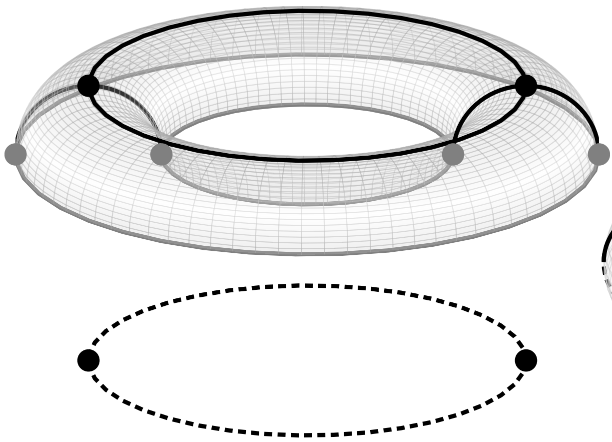
$$\mathbb{T}^2 \sim D^-_0(S^1) \underset{f}{\cup} D^-_1(S^1).$$

Then f maps the boundary $\partial D^-_1(S^1)$ into $D^-_0(S^1)$. By definition, the fibres of $D^-_0(S^1)$ are 0-dimensional. Therefore $D^-_0(S^1) \cong S^1$. The boundary of $D^-_1(S^1)$ is an annulus as shown below. The boundary of this annulus is two copies of S^1, and so the attaching map f maps these to the minimal subcomplex S^1.

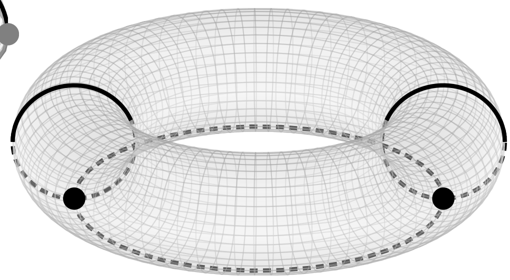
We give the maximal subcomplex the CW-structure of two 0-cells and two 1-cells. Using the disk bundle over this circle, attach two 1-cells via the two 0-cells, and then attach two 2-cells again using the disk bundle. This gives the annulus, as shown in Figure 4a. When attached to the lower circle, the opposite 1-cells and 0-cells of the boundary of the annulus are identified, giving the cell structure shown in Figure 4b.

Therefore we have

$$D_0^-(S^1) \underset{f}{\cup} D_1^-(S^1) \cong e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^1 \cup e^1 \cup e^2 \cup e^2.$$



(a) The upper and lower disk bundles.



(b) The resulting cell structure.

Figure 4

6.3 Morse-Bott inequalities

There are analogous inequalities to the Morse equalities for Morse-Bott functions. We will state and prove these using polynomials as for the Morse inequalities, however the method is slightly different. We follow the construction and proof in [3].

Definition 6.11 (Relative Poincaré polynomial). The relative Poincaré polynomial for X, Y compact spaces, is defined as

$$P_{X,Y}(t) := \sum_k b_k(X,Y)t^k = \sum_k dim\ H_k(X,Y)t^k.$$

Here the Betti numbers come from the relative homology of X with Y.

Lemma 6.12. *If a compact space X has a filtration $X_1 \subset X_2 \subset ... \subset X_n = X$ with X_i closed $\forall i \in \{1, ..., n\}$, then*

$$\sum_{i=1}^{n} P_{X_i, X_{i-1}}(t) = P_{X,\mathbb{F}}(t) + R(t) \tag{11}$$

where $R(t)$ is a polynomial with non-negative coefficients.

Theorem 6.13 (Thom Isomorphism Theorem). *For an $\mathbb{F}$-orientable vector bundle $\pi : E \to X$ of rank r over X, we have*

$$H_{k+r}(D(E), \partial D(E)) \cong H_k(X)$$

over $\mathbb{F}$.

Proof: for X compact. The Thom Isomorphism theorem may be proved by induction. First, suppose that $\pi : E \to X$ is the trivial vector bundle $E = X \times \mathbb{R}^n$. Therefore, we have

$$T(E) = X \times D^n / X \times S^{n-1}.$$

The projection $p : X \mapsto pt$ gives the following map

$$\tau : T(E) = X \times D^n / X \times S^{n-1} \to D^n / S^{n-1} \cong S^{n-1}.$$

Then suppose $a \in H^n(T(E))$ is the image of a generator in cohomology from the map

$$H^n(S^n) \cong \mathbb{Z} \xrightarrow{\tau^*} H^n(T(E)).$$

Taking the cup product with the class $[a]$ gives

$$H^k(X) \xrightarrow{\cup} H^{n+k}(T(E)) = H^{n+k}(X \times D^n, X \times S^{n-1}) = H^{n+k}(X \times S^n, X \times pt)$$

by excision. This map is an isomorphism, by the universal coefficient theorem.

Now suppose that $X = X_1 \cup X_2$ with X_1, X_2 open, and we know the theorem holds for E_1 and E_2, the restrictions of the vector bundle $\pi : E \to X$ to the spaces X_1 and X_2

respectively. Then the theorem also holds for $E_{1,2}$, the restriction to $X_1 \cap X_2$. We can apply Mayer Vietoris for cohomology. We have the sequence

$$... \to H^{k-1}(T(E_{1,2})) \to H^k(T(E)) \to H^k(T(E_1)) \oplus H^k(T(E_2)) \to H^k(T(E_{1,2})) \to ...$$

All the terms involving E_1, E_2 are zero for $k < n$, and therefore by exactness, $H^k(T(E)) = 0$.

For $k = n$, we have

$$H^n(T(E_1)) \cong H_n(T(E_2)) \cong H_n(T(E_{1,2})) \cong \mathbb{Z}.$$

We also have $H^{n-1}(T(E_{1,2})) = 0$, so by exactness, $H^n(T(E)) \cong \mathbb{Z}$. There is a class $[a] \in H^n(T(E))$ mapping to the direct sum of Thom classes in $H^n(T(E_1)) \oplus H^n(T(E_2))$.

In addition, there is the Mayer Vietoris sequence

$$... \to H^{k-1}(X_1 \cap X_2) \to H^k(X) \to H^k(X_1) \oplus H^k(X_2) \to H^k(X_1 \cap X_2) \to ...$$

which maps to the corresponding sequence relating to the Thom spaces by taking the cup product with the Thom classes. We consider the case for $k \geq n$. We know by assumption that the map between the two sequences is an isomorphism for terms involving X_1 and X_2. Therefore by the Five Lemma, it is also an isomorphism on $H^*(X)$. Finally, if X is a finite union of open sets X_i, the theorem may be proved by induction using that the restrictions to each X_i are trivial as before. [17] □

Definition 6.14 (Morse-Bott polynomial). As for Morse functions, there is an associated Morse-Bott polynomial

$$B_f(t) := \sum_{C_i} t^{\lambda_{C_i}} P_{C_i,\mathbb{F}}(t) \tag{12}$$

where $P_{C_i,\mathbb{F}}(t)$ is the Poincaré polynomial for the critical submanifold C_i and λ_{C_i} the index of the critical submanifold C_i. [3]

In fact, the Morse-Bott polynomial reduces to the Morse polynomial in the case of a Morse function.

The Morse-Bott inequalities can be formulated as follows in terms of the Morse-Bott and Poincaré polynomials similarly to the polynomial Morse inequalities used in the proof of the Morse inequalities.

Theorem 6.15 (Morse-Bott Inequalities). *Let $f : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse-Bott function on a compact manifold M. Then*

$$B_f(t) = P_{M,\mathbb{F}}(t) + (1+t)R(t)$$

where $R(t)$ is a polynomial in t with non-negative coefficients. [3]

Proof. First, we define a series of midpoints $t_k := \frac{c_k + c_{k+1}}{2}$ where c_i are the critical values of f and $k \in \{1, ..., s-1\}$, and set $t_0 := c_0 - 1,\ t_s := c_s + 1$.

Then by applying (10) to Lemma 6.12, we get that

$$\sum_{C_i} P_{D^-(C_i),\partial D^-(C_i)} = \sum_k P_{M_k, M_{k-1}} = P_{M,\mathbb{F}}(t) + (1+t)R(t).$$

with $R(t)$ a polynomial with non-negative coefficients. Here, M_k is the set $\{f \leq t_k\}$, and on the right is a sum over all critical submanifolds of f. This is because the Poincaré polynomial $P_{M_k, M_{k-1}}$ has its coefficients which are Betti numbers, derived from the homology groups. Then (10) gives an isomorphism on homology to the homology of the corresponding disk bundles.

The *Thom isomorphism theorem* implies that

$$P_{D(E),\partial D(E)}(t) = t^r P_X(t)$$

where r is the rank of a real vector bundle $\pi : E \to X$ over a compact CW-complex X. This tells us that

$$P_{D^-(C_i),\partial D^-(C_i)} = t^{\lambda_{C_i}} P_{C_i}(t).$$

In particular, the Morse-Bott polynomial is the sum of the Poincaré polynomials $P_{D^-(C_i),\partial D^-(C_i)}(t)$, so

$$B_f(t) = \sum_{C_i} P_{D^-(C_i),\partial D^-(C_i)}(t) = P_{M,\mathbb{F}}(t) + (1+t)R(t)$$

for some $R(t)$ with non-negative coefficients. [3] $\square$

7 Lens Spaces

7.1 Definition

Definition 7.1 (Lens Space). Let $S^3 = \{z_1, z_2 \in \mathbb{C} : z_1^2 + z_2^2 = 1\}$ be the 3-sphere, and p, q be two coprime integers. The action of $\mathbb{Z}/p$ on S^3 is generated by the homeomorphism

$$(z_1, z_2) \mapsto (e^{2i\pi/p} \cdot z_1, e^{2i\pi q/p} \cdot z_2)$$

Then a *3-dimensional Lens Space* $L^3(p; q)$ is the quotient space of S^3 by the action of $\mathbb{Z}/p$.

In general, for $S^{2n-1} = \{(z_1, z_2, ..., z_n) \in \mathbb{C}^n : \sum_{i=1}^n z_i^2 = 1\}$, the action of $\mathbb{Z}/p$ on S^{2n-1} is generated by the homeomorphism

$$(z_1, ..., z_n) \mapsto (e^{2i\pi q_1/p} \cdot z_1, e^{2i\pi q_2/p} \cdot z_2, ..., e^{2i\pi q_n/p} \cdot z_n)$$

for $q_1, q_2, ..., q_n, p$ coprime. Then the *(2n-1)-dimensional Lens Space* $L^n(p; q_1, ..., q_{n-2})$ is the quotient space of S^{2n-1} by the action of $\mathbb{Z}/p$. [22]

Note that this definition gives only odd-dimensional Lens spaces.

In the 3-dimensional case, there is an alternative definition in terms of the torus. The space $L(p; q)$ is equivalent to two solid tori T_1 and T_2 glued together via a homeomorphism sending the meridian of T_1 to a curve on ∂T_2 that runs p times along a longitude and q times along a meridian.

The condition that $gcd(p, q_1, ..., q_n) = 1$ means that the given action of $\mathbb{Z}/p$ on S^{2n-1} is a free action and therefore the resulting quotient space is indeed a manifold.

Lens Spaces are of particular interest because they exhibit some interesting properties. For example in 3 dimensions, the spaces $L^3(7; 1)$ and $L^3(7; 2)$ are homotopy equivalent but not homeomorphic.

7.2 A Morse-Bott Function on a Lens Space

There is a natural Morse-Bott function on a Lens space, coming from a simple function on S^{2n-1} with the relevant identifications. The function on $\mathbb{CP}^n$ that was discussed earlier and also in Milnor's Morse theory [6] can also be applied to the Lens space. On $\mathbb{CP}^n$, the function is a Morse function and has n critical points. However, the same function applied to the Lens space is in fact not a Morse function, but a Morse-Bott function.

For the coordinates in the above definition of a Lens space, the function

$$f(z_1, z_2, ..., z_n) = \sum_{j=1}^{n} c_j |z_j|^2$$

with $c_i \in \mathbb{R}$, is a Morse-Bott function on the n-dimensional Lens space.

In particular, let us consider a 3-dimensional Lens space $L^3(p, q)$. Then we have the function

$$f(z_1, z_2) = c_1|z_1|^2 + c_2|z_2|^2.$$

Similarly to the case for $\mathbb{CP}^n$, we have a local representation

$$f = c_1 + (c_2 - c_1)(x_2^2 + y_2^2).$$

Let us try to find the critical submanifolds for this function. In the local coordinates (z_1, z_2), f can be written in terms of z_2 only by the implicit function theorem as long as $z_1 \neq 0$.

Write $|z_1|\frac{z_2}{z_1} = x_2 + iy_2$, then $|z_1|^2 = 1 - (x_2^2 + y_2^2)$, and f can be rewritten as

$$f = c_1 + (c_2 - c_1)(x_2^2 + y_2^2).$$

Hence

$$df = (2x_2(c_2 - c_1), 2y_2(c_2 - c_1)) = 0 \iff x_2 = y_2 = 0 \quad i.e. \;\; z_2 = 0.$$

So for a critical point, we have $z_2 = 0$, and since the Lens space is a quotient space of the sphere S^3, we have $|z_1|^2 + |z_2|^2 = 1$. Therefore we must have $|z_1|^2 = 1$ and one critical submanifold is the set of points

$$C_1 := \{(z_1, 0) : |z_1|^2 = 1\} \cong S^1.$$

Similarly, taking $z_2 \neq 0$, we get a second critical submanifold

$$C_2 := \{(0, z_2) : |z_2|^2 = 1\} \cong S^1.$$

To see what the indices for these critical manifolds are, we check the Hessian matrices:

$$H_{p \in C_1} = \begin{bmatrix} 2(c_2 - c_1) & 0 \\ 0 & 2(c_2 - c_1) \end{bmatrix}, \qquad H_{p \in C_2} = \begin{bmatrix} 2(c_1 - c_2) & 0 \\ 0 & 2(c_1 - c_2) \end{bmatrix}.$$

Either $2(c_2 - c_1)$ or $2(c_2 - c_2)$ will be negative, meaning one of C_1, C_2 will have index 0 and the other index 2.

We can check that this function is indeed a Morse-Bott function by checking the necessary conditions.

- *Each connected component of the set of critical points $Cr(f)$ is compact*:
  This is true since each connected component is a copy of S^1,
- $T_pS^1 = ker\ H_{f,p}\ \ \forall p \in S^1$ *for each copy of* $S^1 \in Cr(f)$:
  This can be shown by the Hessian matrix directly:

  $$H_{f,p} = \begin{bmatrix} 2(c_2 - c_1) & 0 \\ 0 & 2(c_2 - c_1) \end{bmatrix}$$

  The Hessian is clearly nondegenerate.

8 Morse-Bott Homology

There is more than one method to achieve Morse-Bott homology, often with some more difficulty than in the Morse case. For example, in [23], Hurtubise compares three different methods of computing homology from a Morse-Bott function. Other methods involve constructing other chain complexes. One can find local Morse functions defined on the critical submanifolds of the Morse-Bott function and use these to define a chain complex. Alternatively, there is a method that uses the spectral sequence that comes from the filtration of the manifold M as a result of the Morse-Bott function. [16]

8.1 Perturbing Morse-Bott to Morse

One method is to simply perturb the Morse-Bott function to a Morse function, and then calculate the homology as described earlier for a Morse function using the Morse-Smale-Witten chain complex. [24]

Theorem 8.1. *Let M be a finite dimensional manifold, and $g : M \to \mathbb{R}$ a smooth function. Then $\forall \epsilon > 0$, there is a Morse function $f : M \to \mathbb{R}$ such that for $x \in M$,*

$$sup\{|g(x) - f(x)|\} < \epsilon.$$

That is, this perturbation method is always possible in the finite dimensional case. [25]

It is possible to perturb a Morse-Bott function f to a Morse function by choosing Morse functions g_i defined on the critical submanifolds C_i of f. Then, each function g_i is extended to a tubular neighbourhood T_i of C_i such that g_i is constant in directions normal to C_i. One then defines a local function μ_i that has constant value 1 in some larger tubular neighbourhood $\hat{T}_i$ of C_i, and is zero outside T_i. The function μ_i is constant in the direction parallel to C_i, and decreases smoothly from 1 to 0 on the space $\hat{T}_i \setminus T_i$. This type of function is sometimes called a bump function.

Then, for some small $\epsilon > 0$, one has a function

$$g := f + \epsilon \sum_i \mu_i g_i.$$

The function g is a smooth function, it coincides with f outside a neighbourhood of the critical submanifolds, and near the critical submanifolds it is a Morse function. Therefore g is a Morse function on M, and in fact also a Morse-Smale function.

Notice that we then have

$$Cr(g) = \bigcup_i Cr(g_i).$$

Lemma 8.2. *Suppose the critical submanifold C_i of the Morse-Bott function f has index λ_{C_i}, and also that $p \in C_i$ is a critical point of g_i with index λ_p^i. Then p is also a critical point of g, and*

$$index(p) =: \lambda_p^g = \lambda_{C_i} + \lambda_p^i.$$

[25]

It is now possible to determine the Morse homology of M from the Morse-Smale function $g : M \to \mathbb{R}$.

Example 8.3 (The torus). Consider again the height function on $\mathbb{T}^2$ as a Morse-Bott function with two critical submanifolds, a maximum and minimum circle. Then we may define a Morse function on each of the submanifolds and extend this to one function on $\mathbb{T}^2$ such that away from the two critical circles, we still have the original Morse-Bott function.

This has the effect of tilting both copies of S^1 and applying to them a height function, giving a maximum and minimum point for each. Then, we have a Morse function on $\mathbb{T}^2$ with four critical points, and the Morse homology may be determined in the usual way.

8.2 A filtration from the Morse-Bott function

CW Homology can be very useful for relatively easily determining the Homology of a space with a known filtration as a CW complex. Instead of using the n-cells of a manifold M, it may be possible to use the filtration of the manifold M and its critical submanifolds coming from a Morse-Bott function in order to find the homology.

The 3-dimensional Lens space is again an interesting example to demonstrate this. There is the possibility of then generalising to higher dimensional Lens spaces.

Example 8.4 (3-dimensional Lens space). Let $L := L(p, q)$ be a Lens space. Then, as determined in the previous chapter, the critical submanifolds of the Morse-Bott function f on L are two copies of S^1, one of index 0 and one of index 2.

Consider the filtration $\emptyset \subset S^1 \subset L$ where we S^1 is the maximal critical submanifold of dimension 2. Then from the short exact sequence of the pair, we get the induced long exact sequence on homology:

$$\begin{aligned}&0 \to H_3(L) \to H_3(L, S^1) \to H_2(S^1) \to H_2(L) \to H_2(L, S^1) \to H_1(S^1) \to \\ &\to H_1(L) \to H_1(L, S^1) \to H_0(S^1) \to H_0(L) \to 0\end{aligned}$$

We know the homology of S^1:

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & otherwise \end{cases}$$

We can also simplify further by using the Thom Isomorphism theorem and Theorem 6.9. The negative normal bundle of the maximal critical submanifold S^1 in $L(p; q)$ has rank 2 over S^1. That is, the subspace of the normal bundle of C_2 on which the Hessian is negative definite is 2 by definition of the index. This implies that

$$H_n(L, S^1) \cong H_{n-2}(S^1).$$

So we now have:

$$0 \to H_3(L) \to H_3(L, S^1) \cong H_1(S^1) \cong \mathbb{Z} \to H_2(S^1) = 0 \tag{13}$$

$$0 \to H_2(L) \to H_2(L, S^1) \cong H_0(S^1) \cong \mathbb{Z} \xrightarrow{\delta} H_1(S^1) \cong \mathbb{Z} \to H_1(L) \to H_1(L, S^1) \cong 0 \tag{14}$$

$$0 \to H_0(S^1) = \mathbb{Z} \to H_0(L) \to 0 \tag{15}$$

Hence, by exactness in (13) we see that $H_3(L) \cong \mathbb{Z}$, and similarly, in (15) we get that $H_0(L) \cong \mathbb{Z}$.

The map $\delta : H_2(L, S^1) \to H_1(S^1)$ in (14) is multiplication by p. This is because the boundary of the disc generating $H_2(L, S^1)$ wraps around the copy of S^1 p times when mapped into $H_1(S^1)$, due to the identifications of the Lens space. Thus, we get by exactness that $H_1(L) \cong \mathbb{Z}/p$ and $H_2(L) = 0$.

Therefore, we have determined all the homology groups of the 3-dimensional Lens space:

$$H_k(L(p; q), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 3 \\ \mathbb{Z}/p & k = 1 \\ 0 & otherwise. \end{cases}$$

It is not too difficult to generalise the above example to the n-dimensional case. A little more consideration is required to determine the associated boundary maps.

Example 8.5 (m-dimensional Lens space). Let L^m be the m-dimensional Lens space $L(p; q_1, q_2, ...q_n)$, where $m = 2n - 1$. Then by the same method as used for the 3-dimensional Lens space, L^m has critical submanifolds given by

$$C_k = \{(0, ..., 0, z_k, 0, ..., 0) : |z_k|^2 = 1\} \cong S^1, \quad \forall k \in \{1, ..., n\},$$

with even indices $0, 2, ..., 2n - 2$, or equivalently $0, 2, ..., m - 1$.

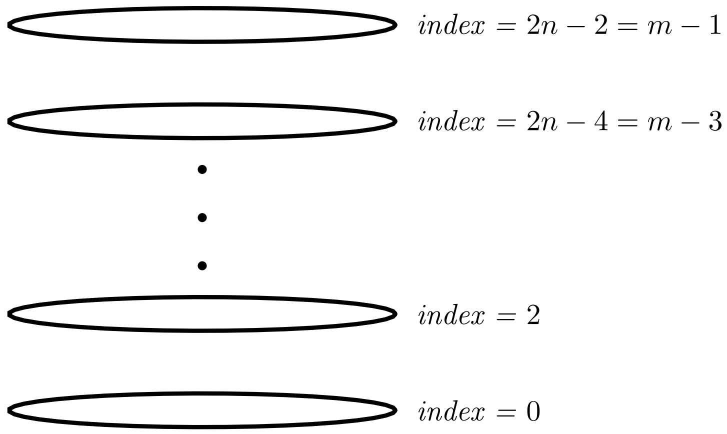


Figure 5: Ordering of the critical submanifolds.

Then we have a filtration

$$\emptyset \subset S^1 \subset L^3 \subset L^5 \subset ... \subset L^m.$$

We have the following long exact sequence in homology.

$$0 \to H_m(L^m) \to H_m(L^m, L^{m-2}) \to H_{m-1}(L^{m-2}) \to H_{m-1}(L^m) \to \ \ldots \ \to H_2(L^m)$$
$$\to H_2(L^m, L^{m-2}) \to H_1(L^{m-2}) \to H_1(L^m) \to H_1(L^m, L^{m-2}) \to H_0(L^{m-2}) \to H_0(L^m) \to 0.$$

We can see that the homotopy type of L^{m-2} is the same as that of L^m_{m-3}, the sublevel set of points up to and including the index $m - 3$ submanifold in L^m. Since we do not pass any critical values, this has the same homotopy type of L^m_{m-2}.

Therefore we have that

$$H_k(L^m_{m-2}) \cong H_k(L^{m-2}).$$

The manifold L^m has an extra critical submanifold, a copy of S^1 with index $m - 1$ that L^{m-2} does not. This can be explained by the fact that $L^{m-2} \cong L^m_{m-2}$. Therefore L^m and L^m_{m-2} can replace $M^{c+\epsilon}$ and $M^{c-\epsilon}$ in the isomorphism (10) in theorem 6.9. The rank of the negative normal bundle E_- of the maximum critical submanifold S^1 over S^1 is $m-1$, since the maximal critical submanifold has index $m-1$ and hence the maximal subspace of the normal bundle $E_-(S^1)$ on which the Hessian has negative eigenvalues has rank $m-1$. That is, the rank of E_-.

Therefore, by theorem 6.9 and the Thom isomorphism theorem, we have

$$\begin{aligned} H_k(D^-(E), \partial D^-(E)) &\cong H_k(L^m, L^{m-2}) \\ &\cong H_{k-(m-1)}(S^1). \end{aligned}$$

Substituting the known homology of S^1 into the long exact sequence above, we obtain the following exact sequences.

$$0 \to H_m(L^m) \to H_m(L^m, L^{m-2}) \cong H_1(S^1) \cong \mathbb{Z} \to H_{m-1}(L^{m-2}) = 0 \tag{16}$$

$$\begin{aligned} &0 \to H_{m-1}(L^m) \to H_{m-1}(L^m, L^{m-2}) \cong H_0(S^1) \cong \mathbb{Z} \to H_{m-2}(L^{m-2}) \cong \mathbb{Z} \to \\ &\quad \to H_{m-2}(L^m) \to H_{m-2}(L^m, L^{m-2}) = 0 ... \end{aligned} \tag{17}$$

$$...0 \to H_k(L^{m-2}) \to H_k(L^m) \to H_k(L^m, L^{m-2}) = 0 ... \tag{18}$$

$$...0 \to H_0(L^{m-2}) \cong \mathbb{Z} \to H_0(L^m) \to 0$$

In particular, for all $k < m-2$, we have by exactness in (18) that

$$H_k(L^m) \cong H_k(L^{m-2}).$$

Therefore by induction, we have so far that

$$H_k(L^m) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/p & k \text{ } odd, k < m-2 \\ 0 & k \text{ } even, k < m-2 \end{cases}$$

Similarly, by exactness in (16), we get that $H_m(L^m) \cong \mathbb{Z}$.

It remains to consider in (17) the map

$$\delta : H_{m-1}(L^m, L^{m-2}) \cong \mathbb{Z} \to H_{m-2}(L^{m-2}) \cong \mathbb{Z}.$$

We have the relationship

$$H_{m-1}(L^m, L^{m-2}) \cong H_{m-1}(D^m, S^{m-2}) \cong H_{m-2}(S^{m-2})$$

by excision and the following long exact sequence of the pair.

$$0 = H_{m-1}(S^{m-2}) \to H_{m-1}(D^m, S^{m-2}) \to H_{m-2}(S^{m-2}) \cong \mathbb{Z} \to H_{m-1}(D^m) = 0.$$

Thus, we have a map

$$\delta' : H_{m-2}(S^{m-2}) \to H_{n=m-2}(L^{m-2}),$$

with $\delta \simeq \delta'$. The map δ' is induced by some homeomorphism f between two manifolds of the same dimension, S^{m-2} and L^{m-2}. Notice that S^{m-2} is the universal cover for any $(m-2)$-dimensional Lens space L^{m-2}.

The quotient map by the action of $\mathbb{Z}/p$ maps S^{m-2} smoothly to L^{m-2}. By the identifications

$$(z_1, ..., z_n) \sim (e^{2i\pi q_1/p} \cdot z_1, e^{2i\pi q_2/p} \cdot z_2, ..., e^{2i\pi q_n/p} \cdot z_n),$$

we can see that the preimage of a point y in L^{m-2} under a smooth map from S^{m-2} is a set of p points $(x_1, ..., x_p)$. The local degree at each x_i is ± 1. By choosing an orientation, we can take this to be 1, and therefore we have that the degree of the map is p.

Thus, we have that δ is multiplication by p, giving us the following sequence,

$$0 \to H_{m-1}(L^m) \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to H_{m-2}(L^m) \to 0.$$

By exactness, this gives $H_{m-1}(L^m) = 0$ and $H_{m-2}(L^m) \cong \mathbb{Z}/p$.

Therefore, we have obtained the homology groups for any Lens space $L^m(p, q_1, ..., q_n)$.

$$H_k(L^m) = \begin{cases} \mathbb{Z} & \text{k}{=}0, m \\ \mathbb{Z}/p & \text{k} < m \textit{ and odd} \\ 0 & \text{otherwise} \end{cases} \iff H_k(L^{2n-1}) = \begin{cases} \mathbb{Z} & \text{k}{=}0, 2n-1 \\ \mathbb{Z}/p & \text{k} < 2n-1 \textit{ and odd} \\ 0 & \text{otherwise.} \end{cases}$$

9 Flow Categories for Morse-Bott functions

The method of using flow categories for Morse functions to calculate Morse homology is well established. It is interesting to consider whether a similar approach could be applied to Morse-Bott functions. In this case, the notion of a flow category needs to be re-defined in order to make sense for the Morse-Bott situation.

9.1 The Flow Category

First, we will introduce some basic category theory and define the standard flow category.

Definition 9.1 (Category). A *Category*, $\mathcal{C}$, is a collection of *Objects* and *Morphisms* between the objects, such that the following are satisfied:

- for $A, B, C \in ob(\mathcal{C}), f : A \to B,\ g : B \to C$, there exists a composition map $g \circ f : A \to C$,
- for $A, B, C, D \in ob(\mathcal{C}),\ f : A \to B,\ g : B \to C,\ h : C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$,
- $\exists$ an identity $1_A : A \to A \ \in mor(\mathcal{C}) \quad \forall A \in ob(\mathcal{C})$.

That is, each object has an identity morphism, and all morphisms in the category satisfy the associativity law. For a function $f : A \to B$, it is sometimes written $f \in \mathcal{C}(A, B)$. We also write $dom(f)$ and $cod(f)$ to mean the domain and codomain of f respectively. The objects of a category can equivalently be thought of as sets, and the morphisms as functions. [26]

Definition 9.2 (Topological Category). A *topological category* is a category where the sets Mor and Ob are topological spaces, and the identity, composition, domain and codomain maps are all continuous. [14]

The *Flow category* is another example of a category, which is important in Morse theory.

Definition 9.3 (Flow Category). Let $f : M \to \mathbb{R}$ be a Morse function on a manifold M. Then a category $\mathcal{C}_f$ with

$$\begin{aligned} ob(\mathcal{C}_f) &= \{\text{critical points of } f\}, \\ mor(\mathcal{C}_f)(A, B) &= \{\text{piecewise gradient trajectories from } A \text{ to } B\} \end{aligned}$$

for $A, B \in ob(\mathcal{C}_f)$ is called the *flow category* associated to f. Composition of morphisms is concatenation of the trajectories. [14]

The set $ob(\mathcal{C}_f)$ is in fact a topological space, as a subspace of M. It has the discrete topology since all the critical points of a Morse function are isolated and hence so are the neighbourhoods. It can further be shown that the flow category is also a topological category. [14]

9.2 A Flow Category defined for Morse-Bott Functions

In a paper [7] by Cohen et al, they define a flow category for a Morse-Bott function in a way that there is a strong connection to the homology of the manifold M. In this section, this method will be applied to the example of a Lens space.

The flow category for a Morse function has the critical points as the objects, and the gradient flow between these points as the morphisms.

For a Morse-Bott function f, suppose the objects are the critical submanifolds for a given critical value, and the morphisms the piecewise gradient flows between any pair of these. That is, a flow line γ from p to q may stop at other critical points between p and q. We have γ as the concatenation of p curves,

$$\gamma := \gamma_r \circ \gamma_{r-1} \circ \ldots \circ \gamma_0,$$

where

$$\lim_{t\to-\infty} \gamma_0(t) = q, \qquad \lim_{t\to\infty} \gamma_p(t) = p$$
$$\lim_{t\to-\infty} \gamma_i(t) = p_i, \quad \lim_{t\to\infty} \gamma_i(t) = p_{i+1}$$

for p_i critical points with $p_0 = q$ and $p_{r+1} = p$. i.e. each γ_i is a flow line from p_i to p_{i+1}.

When the Smale-transversality condition is satisfied by the gradient flow, which in simple cases it is, we may define the flow category as follows. If t_i are the critical values associated to the critical submanifolds C_i, and assume that $t_0 > t_1 > \ldots > t_n$. Then define the object set of the flow category $\mathcal{C}_f$ as the collection of critical submanifolds

$$Ob(\mathcal{C}_f) := \coprod C_i.$$

Then, the spaces of Morphisms C_{ij} from C_i to C_j are compact manifolds with corners, and the category $\mathcal{C}_f$ is a topological category.

9.3 Flow Category on a Lens Space

Example 9.4. As we found earlier, we have two critical submanifolds for the function

$$f(z_1, z_2) = c_1|z_1|^2 + c_2|z_2|^2$$

on the 3-dimensional Lens space.

To find the gradient flow between these to critical manifolds, we can make a 'slice' in between the two critical levels by taking a preimage of a value somewhere within this region. Away from the critical submanifolds, we have that $z_0, z_1 \neq 0$, so we may express f in terms of either z_0 or z_1.

Without loss of generality, suppose we can write

$$f = c_1 + (c_2 - c_1)(x_2^2 + y_2^2)$$

Then, for example, calculate

$$\begin{aligned} f^{-1}\left(\frac{1}{2}\right) &= \left\{(x_1 + iy_1, x_2 + iy_2) \in L^3(p,q) \ : \ c_1 + (c_2 - c_1)(x_2^2 + y_2^2) = \frac{1}{2}\right\} \\ &= \left\{(z_1, z_2) \in L^3(p,q) \ : \ |z_2|^2 = \frac{1}{c_2 - c_1}\left(\frac{1}{2} - c_1\right)\right\}. \end{aligned}$$

This gives us $\{(z_1, z_2) : |z_2|^2 = c\}$ for some constant c, and since we have $|z_1|^2 + |z_2|^2 = 1$, we also have that $|z_1|^2 = 1 - c$ and so the resulting space is $S^1 \times S^1$, i.e. the Torus $\mathbb{T}^2$.

The formulation of Morse-Bott flow categories comes from [7], and the example follows closely to the $\mathbb{CP}^n$ example from [6].

Lens spaces can exhibit unusual properties in that there are some pairs of Lens spaces with the same homology groups and homotopy groups, that are not homotopy equivalent. Others are homotopy equivalent, however not homeomorphic.

It may be possible to use the knowledge of the flow between the two critical submanifolds on the Lens Space and an attaching map from this torus of flow lines to the whole space to find the homeomorphism type.

It may also be possible to further study the Lens spaces by considering the moduli spaces of the flow between the critical submanifolds, again this may require careful consideration of the properties of the Lens space and the Morse-Bott function.

If this could be achieved, the results would form a completion of this example. Perhaps for specific cases, it could allow the unusual behaviour of the Lens space regarding the homeomorphism and homotopy types, to be realised through the study of a very simple function.

10 Conclusion

Morse theory has been an invaluable tool in providing fundamental results regarding the structure and topology of a variety of manifolds. It can be extended further to give results for non-compact manifolds, complex spaces, infinite dimensional manifolds [6], and also many areas of pure and applied mathematics.

In general, Morse-Bott functions may have less concrete results, however it has been shown that in some cases one can obtain elegant solutions to problems with the use of a Morse-Bott function. In particular, there are cases where a simple Morse-Bott function on a manifold may arise more naturally than a Morse function. With a simple Morse-Bott function, it can often be possible to determine all the same information about the manifold. In other cases, there is always the possibility to perturb this to a Morse function instead.

While there are many approaches to determining homology, whether it is singular, simplicial or cellular, Morse and Morse-Bott homology are effective alternatives when one has knowledge of a function on a manifold rather than of the structure of the manifold itself.

Since the early stages of Morse theory, using the fundamental properties of critical points to build up the structure of topological spaces, the subject has enabled a huge number of other concepts to be explored. The subject is continuing to grow, and will no doubt lead to many more results in its application.

References

[1] R. Bott. Marston Morse and his mathematical works. *Bulletin (New Series) of the American Mathematical Society*, 3(3), 1980.

[2] R. Bott. Lectures on Morse theory, old and new. *Bulletin (New Series) of the American Mathematical Society*, 7(2), 1982.

[3] L. Nicolaescu. *An Invitation to Morse Theory.* Springer, 2007.

[4] Y. Chen. A brief history of Morse homology. https://pdfs.semanticscholar.org/56d0/1c582bd8475acadfa04cb753356658b4b393.pdf. [Accessed: 30/04/2020].

[5] A. Banyaga and D. Hurtubise. *Lectures on Morse Homology.* Kluwer Academic Publishers, 2004.

[6] J. Milnor. *Morse Theory.* Princeton University Press, 1963.

[7] J. D. S. Jones R. Cohen and G. B. Segal. Floer's infinite dimensional Morse theory and homotopy theory. volume 133, pages 297–325. Birkhäuser Basel, 1995.

[8] P. Tumarkin. Riemannian geometry IV lecture notes, 2020. [Accessed: 30/04/2020].

[9] L. Kinsey. *Topology of surfaces.* Springer, 1993.

[10] A. Dold. *Lectures on Algebraic Topology.* Springer, 1995.

[11] A. Hatcher. *Algebraic topology.* Cambridge university press, 2002.

[12] M. Powell. Algebraic topology IV lecture notes, 2020. [Accessed: 30/04/2020].

[13] P. Harpe. Brouwer degree, domination of manifolds, and groups presentable by products. sep 2016.

[14] T. O. Rot. Morse theory via the flow category. diploma thesis, University of Utrecht, jul 2010.

[15] M. Hutchings. Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves). https://www.math.toronto.edu/mgualt/Morse%20Theory/mfp.pdf, dec 2002. [Accessed: 30/04/2020].

[16] A. Banyaga and D. Hurtubise. Morse-Bott homology. *Transactions of the American Mathematical Society*, 362(8), aug 2010.

[17] R. Cohen. The topology of fiber bundles. https://math.stanford.edu/ ralph/fiber.pdf, aug 1998. [Accessed: 30/04/2020].

[18] C. Wendl. Lecture notes on bundles and connections. https://www.mathematik.hu-berlin.de/ wendl/connections.html, sep 2008. [Accessed: 30/04/2020].

[19] C. T. C. Wall. *Differential Topology.* Cambridge University Press, 2016.

[20] The isotopy extension theorem. http://math.ucr.edu/ res/math260s10/isotopyextension.pdf. [Accessed: 30/04/2020].

[21] M. Guest. Morse theory in the 1990s. https://arxiv.org/abs/math/0104155, apr 2001. [Accessed: 30/04/2020].

[22] C. Wall. *Surgery on compact manifolds.* American mathematical society, 2 edition, 1999.

[23] D. Hurtubise. Three approaches to Morse-Bott homology. *African Diaspora Journal of Mathematics*, 14(2), 2012.

[24] A. Banyaga and D. Hurtubise. Cascades and perturbed Morse-Bott functions. *Algebraic & Geometric Topology*, 13(1), feb 2013.

[25] D. Hurtubise. Morse and Morse-Bott homology. University of Notre Dame, mar 2013. [Accessed: 30/04/2020].

[26] S. Awodey. *Category Theory.* Oxford Logic Guides, Oxford University Press, 2006.

COVID19 IMPACT SHEET
Project 3/4
Department of Mathematical Sciences

Student Name: Leyna Watson May

Year group (3/4): 4

Project Topic: Morse Theory

Project supervisor(s): Dr Dirk Schuetz

Did Covid19 prevent you from completing part of your project report (Yes/No):
No

If 'Yes', please indicate what it prevented you from doing (max 100 words):

Please summarise the action taken in response (max 100 words):