Hidden Symmetries, Trees and Operads

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Abstract

We investigate a number of symmetric structures, including spaces of trees, partition complexes, and operads. In particular, these spaces have natural actions of the symmetric group by permuting leaf labels, elements, and labels of inputs to operations respectively. These also share the property that the action of the symmetric group Σ_n may be extended to an action of a larger symmetric group Σ_{n+1} on arity n-objects. In some cases such as the partition complex, this action is not obvious, and is realised through its relationship with the tree space of Robinson and Whitehouse in [RW96]. In the case of operads, the ability to extend the action in this way gives an extra structure to those for which it is compatible with the operad structure. Operads for which this is possible are called cyclic operads, a concept introduced by Getzler and Kapranov in [GK95].

In this thesis we write explicit proofs of equivalences of skeletal and non-skeletal definitions of cyclic operads and cooperads. We explore extended symmetric group actions in the partition poset, on collections of finite sets, and on examples of operads and cooperads with cyclic structure. We give a topological construction using suspensions of tree spaces, that have a variant of a cyclic structure that we introduce. This leads to an anticyclic structure on the desuspension of the Lie operad.

Contents

Introduction

In this thesis, we explore symmetric group actions on various spaces, in particular 'hidden' actions of Σ_{n+1} that restrict to a Σ_n action.

There are many topological spaces that admit an obvious Σ_n action, for example by permuting elements. In some cases, there is also an action of the symmetric group Σ_{n+1} which is often not an obvious action, and hence 'hidden'. This additional action provides a Σ_{n+1} representation that restricts to the Σ_n representation. An example of a space where both Σ_n and Σ_{n+1} have relatively simple actions is a space of trees, as described in [RW96].

The subject of operads, and in particular cyclic operads, relates somewhat naturally to the concept of hidden symmetric group actions. We may construct operad and cooperad structures from the spaces that motivated this project, and use such actions to give more structure on these (co)ooperads.

The topics of this thesis are essentially split into two halves, the first of which is operads and cooperads, and the second of which is the tree space and partition complex. These are tied together in the final chapter, where we give our main results. We will give an overview of the relevant literature, and motivation for each of these.

Operads and cooperads

The term 'operad' was first introduced and rigorously defined in 1972 by May in [May72], although the concept was present in the work of authors including Boardman, Vogt and Stasheff in the 1960s. Operads encode sets of n-ary operations, with composition and other nice properties. While they were initially developed with the purpose of studying iterated loop spaces and algebras, applications now extend to many areas of mathematics. These include homotopy theory, low dimensional topology, category theory, algebraic geometry and string theory.

In particular, symmetric operads have a natural action of the symmetric group Σ_n on each set of n-ary operations by permuting the inputs, whereas non-symmetric operads do not have such actions. Symmetric operads are more commonly used, because having the ability to permute inputs gives useful structure to the operad and its algebras.

Since the main purpose of this thesis is to study actions of symmetric groups, we mainly focus on symmetric operads. Our proofs of equivalences of definitions apply to symmetric operads, and while we include one or two non-symmetric operad examples for illustrative purposes, the main operads of importance that we study will be symmetric.

Cyclic operads were introduced by Getzler and Kapranov, defined in [GK95], as a way of calculating cyclic homology and to generalise the theory of cyclic algebras. In a (symmetric) cyclic operad, one has an extension of the Σ_n action on the collection of n-ary operations to an action of the bigger group Σ_{n+1}. This action is subject to compatibility conditions, which encode equivariance for the extended action. Many well-known operads turn out to have this property, such as the associative, commutative, and Lie operads.

Cooperads are a dual notion to operads, in particular a cooperad in a symmetric monoidal category $\mathcal{C}$ is an operad in the opposite category $\mathcal{C}^{op}$. They can also be defined in the same way as operads, however with a 'cocomposition' operation instead of a composition operation. As operads are an important tool in studying algebras, cooperads are a natural notion in that they are used to study coalgebras.

It is then natural to define cyclic cooperads in exactly the same way as for operads, and the cooperads with this property characterise additional structure on certain coalgebras.

The tree space and the partition complex

In [RW96], Robinson and Whitehouse introduce spaces $\widetilde{T}_n$ of n-trees, giving their homotopy type, and exploring symmetric group action on the spaces. These spaces are given by non-planar trees with n leaves and labels in $\{1, 2, \ldots, n\}$ as well as a root. The trees are also weighted with lengths applied to internal edges, and the ability to continuously shrink such edges results in these spaces being cubical (or simplicial) complexes that are homotopy equivalent to a wedge of spheres.

There is a natural action of Σ_n on the spaces of trees by permuting the labels of the leaves, and Robinson and Whitehouse show that the character of the associated representation of Σ_n is that of the Lie representation. In their paper, they define the Tree representation of Σ_{n+1} on the space T_n of fully grown trees using the natural action of Σ_{n+1} that permutes the leaf labels of the trees as well as the root label.

These tree spaces are shown by Robinson in [Rob04] to be homeomorphic to the partition complexes Λ_n. In particular, the complex Λ_n is a geometric realisation of the nerve of the poset of partitions with ordering given by refinement. This partition complex appears in numerous places in the literature. It is related to the Goodwillie tower of the identity functor, and this relationship comes from the work of Johnson in [Joh95], and Arone and Mahowald in [AM99].

Structure of thesis

Chapter 1: Background

This first half of this chapter consists of general background definitions and concepts that will be referred to throughout the thesis. These include results in areas such as group theory, category theory and set theory.

The second half of the chapter is dedicated to the results in representation theory, with a particular focus on representations of the symmetric group. Representation theory is an integral part of this thesis, as it is the tool used to distinguish actions of the symmetric group from one another, and explore their properties. We use it when discussing explicit actions on spaces of trees and partitions, and also when discussing the structure of certain operads.

We also set out a number of conventions that will be used throughout the thesis, and introduce notation.

Chapter 2: Operads

In this chapter we introduce operads, a subject that forms a large portion of this thesis. We give two main definitions that we compare in detail, with the purpose of extending to settings with additional structure in the following chapters. In particular, we write a proof of the equivalence of skeletal and non-skeletal definitions of operads that was first proved by Lukács in [Luk10] and [Luk13]. Our proof expands on the detail and is structured differently from that of Lukács, with additional clarification and discussion of technicalities such as the use of the disjoint union, and the renumbering map.

We set up notation and introduce the necessary tools that we will use to define operads. This consists of the deleted disjoint union of pointed finite sets, the renumbering map that is intrinsic to skeletal operad definitions, and the composition of permutations and bijections that act on operations in an operad.

We then give the skeletal definition followed by the non-skeletal definition of a symmetric operad, using partial composition operations $\bullet_i$, or in the non-skeletal case $\circ_x$, and define the categories of such operads.

The main purpose of this chapter is the proof of the equivalence of the skeletal and non-skeletal definitions. We do this by giving an equivalence of the operad categories, by lifting the extension and restriction functors between the symmetric groupoid and the category of finite sets and bijections to the operad categories, as Lukács does. This result is given in Theorem 2.4.2.4.

The remaining sections are dedicated to a brief discussion of non-symmetric operads, some key running examples of operads that we will refer back to in later chapters, and some useful structures and properties of operads. The main examples are the associative operad and the Lie operad, both of which are relevant to operads of trees that are discussed in later chapters. We also define operad algebras, and operad (co)homology. These are some of the most important structures that operads are used to study, and operad homology will be of particular importance in Chapter 6.

Chapter 3: Cyclic operads

Cyclic operads are a collection of operads that have additional structure given by an extension of the symmetric group action in the symmetric case, or simply a cyclic group action in the non-symmetric case. As we did in Chapter 2, we look at the skeletal and non-skeletal definitions of cyclic operads, and we prove the equivalence of these definitions by checking the same proof holds for the extra structure of these operads. In particular, we use the definition of Markl in [Mar99] that is in the form of an operad with additional structure. We note that this definition has one more cyclic condition than the initial definition by Getzler and Kapranov in [GK95]. It is thought that this extra condition may follow in some cases from the other, however this has not been proved in general, and so we include it in our definitions and checks.

The inconvenience of the renumbering map is highlighted in this chapter, and therefore the advantage of being able to pass between skeletal and non-skeletal settings is clear.

First, we specify conventions regarding base points and the ambient categories. We also discuss the renumbering map in this setting, as well as conventions for the extension of the symmetric group action.

We define skeletal cyclic operads using $(n+1)$-cycles as in the definitions of Getzler, Kapranov and Markl, and we define the associated category of skeletal cyclic operads. We then give an alternative definition that uses transpositions instead, following the idea of Obradović in [Obr17]. The details of this method are not in the literature, and some care is needed particularly when dealing with the renumbering map in the skeletal case. The reason for doing this is so the skeletal definition can be easily compared with the non-skeletal definition, since the big cycle doesn't make sense in the case of finite sets where there is no canonical ordering. We show that the definition of skeletal cyclic operads with transpositions is equivalent to that with cycles, and finally we define the skeletal cyclic operad category.

This is followed by the non-skeletal definition of cyclic operads, and the category of such cyclic operads. This definition extends the action of basepoint preserving bijections to bijections that permute the basepoint, by using generating transpositions of the form $(0, x)$.

We prove the equivalence of the skeletal and non-skeletal definitions, by again showing that the categories of cyclic operads are equivalent, following the same structure of the proof for general operads by checking compatibility with the additional action of permutations and bijections. This is the main result of this chapter and is given in Theorem 3.4.0.2.

We discuss cyclic non-symmetric operads, and give the example of cyclically labelled trees, that is related to the construction in chapter 6.

The remaining sections are used to consider the uniqueness of cyclic structure, with the associative operad as a main example, and we give some other examples and non-examples of cyclic operads. We define anticyclic operads, introduced by Getzler and Kapranov in [GK95], which are a modified version of cyclic operads. These will again be relevant in chapter 6.

Chapter 4: Cooperads

In this chapter, we transfer the results from operads and cyclic operads into the opposite setting. We write definitions of cooperads by defining cocomposition operations and the associated axioms which are simply the same as in the operad case with some (not all) of the diagram arrows reversed. While cooperads can be considered as a dual notion to operads, it will be instructive to see the complete definitions in this form for the examples that we study in other chapters. That is, we later construct a cooperad and show it has cyclic structure, by checking its properties against these explicit definitions. Because the definitions are essentially the same aside from having a cocomposition operation rather than a composition operation, we omit most of the detail of the proof to save repeating the proofs in the previous chapters.

We give the skeletal definition and define the skeletal cooperad category. The commutative diagrams of the definition are almost dual to the diagrams in the operad case, however the direction of permutation arrows is not reversed.

The equivalence for cooperads follows automatically from the same argument as for operads and is given in Theorem 4.4.0.1.

We give some examples of cooperads, and briefly discuss cooperad coalgebras and (co)homology. As expected, these are essentially dual to algebras over operads, and operad (co)homology.

Finally, in the second part of the chapter, we repeat the process for cyclic cooperads. We write the skeletal and non-skeletal definitions, and the equivalence given in Theorem 4.8.2.1 follows from the same argument as for cyclic operads.

Chapter 5: Trees and partitions

For now, we leave behind operads and cooperads, and turn our attention to some explicit examples of hidden symmetric group actions in other settings. This chapter is based on the work of Robinson and Whitehouse on a space of trees, and a hidden symmetric group action on this space. This action and the associated representations are the main motivation for the work in this thesis. We define the space, give its important properties, and describe the actions of the symmetric groups as well as the associated representations. This space reappears in Chapter 6 when we construct a cooperad from it.

Robinson shows in [Rob04] that this tree space is homeomorphic to a poset of partitions of sets $\{1, 2, \ldots, n\}$. This means that the action of Σ_{n+1} that is natural in the tree space transfers to an action on the poset of partitions. In this space, the action is not natural, as we have no obvious $(n+1)^{th}$ object to permute.

We begin the chapter by introducing the tree space, followed by the partition poset, giving examples and diagrams for low dimensional cases. We also discuss the key results from the two papers mentioned above regarding the Σ_{n+1} actions on these spaces.

We then explore in more detail what the transferred action looks like in the poset of partitions, and focus on the partitions that correspond to vertices in the tree space. We use a bijection between finite sets and partitions with only one part of size greater than 1 to study explicitly this action as an action on finite sets. We give the representations of Σ_{n+1} associated with these actions, and the representations of Σ_n given by the action that they restrict to in Propositions 5.6.2.1, 5.7.1.3 and 5.7.1.4 respectively.

This motivates some questions about potential hidden actions on finite sets with a fixed size, and the remainder of the chapter is spent exploring this. We use representation theory to show that the various actions of symmetric groups are not equivalent. This result is given in Proposition 5.8.3.2.

Finally, we revisit the poset of partitions and in particular, explore whether the explicit hidden action we have found on partitions of a given shape can be extended to an action on the whole poset. We use the characters of the representations to show that in this case it cannot.

Chapter 6: A cooperad of trees

This chapter contains the main results of the thesis.

We revisit the tree space from Chapter 5 and construct a cooperad structure on suspensions of this space. The motivation for this is work of Ching in his thesis [Chi05], where he constructs a cooperad of trees that is closely related to the spectral Lie operad. It is known that the spectral Lie operad comes from a double suspension of the space of partitions that we define in Chapter 5, where we also discuss that this is homeomorphic to the tree space defined by Robinson and Whitehouse in [RW96]. Ching shows that the Spanier-Whitehead dual of his cooperad of trees gives the Lie operad in spectra, and is also related to the partition complex.

We construct a similar non-counital cooperad denoted $\mathcal{T}$ from the spaces T_n of trees directly, using a suitable quotient to simplify the construction and deal with one of the suspensions. We then show that after suspension and taking the Spanier-Whitehead dual, the resulting operad is in fact equivalent to the Lie operad in spectra for $n \geq 2$.

The key advantage to this method, is that unlike the cooperad constructed by Ching, the spaces T_n do not distinguish the root of the trees. This enables one to freely permute the root label with the other leaf labels, giving an obvious cyclic structure in these spaces. Therefore, by the properties of the suspension and Spanier-Whitehead dual, we are able to show an explicit cyclic structure on the resulting operad constructed in this way, and show that this gives a cyclic structure on the spectral Lie operad.

The cooperad $\mathcal{T}$ has a natural extension of the symmetric group action that almost satisfies the cyclic cooperad conditions, apart from a swap of suspension coordinates. This leads us to introduce a new variant of cyclic structure that we call 'twisted cyclic' structure. This property is specific to topological operads or cooperads for which all the spaces are themselves suspension spaces, and is found in Definition 6.4.0.1.

We show that there is a twisted cyclic structure on $\mathcal{T}$ in Theorem 6.4.0.3. Then we extend the non-unital operad given by the homology of the dual $\mathcal{DT}$ to an operad we call $\widehat{\mathcal{T}}$ that includes $n = 1$, which is introduced in Definition 6.7.0.3. In Theorem 6.7.0.4, we show that our construction gives an operad that is equivalent to the operadic desuspension of the Lie operad.

Finally, in Theorem 6.8.0.1, we show that there is an anticyclic structure on $\widehat{\mathcal{T}}$. This gives anticyclic structure on the desuspension of the Lie operad.

Appendix A

The appendix contains two technical case by case proofs from Chapter 5. First is the proof of Proposition 5.7.0.2. The second is the proof of Proposition 5.8.2.3.

Appendix B: Future work and questions

We discuss some potential questions for future consideration, as well as topics of interest that were studied in this PhD, but were unable to yield results within the time. This includes some thoughts on the relationship between the tree space and configuration spaces, as well as related questions regarding operads constructed from partitions.

Chapter 1

Background

In this chapter we will briefly cover some key results and definitions from various topics that will be needed throughout the thesis. Background that is specific to a particular chapter will be included at the beginning of the relevant chapter. The main areas of interest are groups, group action (focusing mainly on the symmetric group), sets, and category theory.

The second part of the background chapter will be dedicated to representation theory, which is a useful tool and a recurring theme throughout the thesis. As we deal a lot with actions of the symmetric group, we will cover in some detail representations of the symmetric group, which will be essential in describing and differentiating group actions.

1.1 The symmetric group

The symmetric group is the main object of study in this thesis, where we consider its action on spaces and other topological objects.

Definition 1.1.0.1 (The symmetric group)**.** The symmetric group Σ_n on n elements is the group consisting of the permutations of the elements. The group operation is composition of permutations.

We sometimes write permutations using cycle notation. For example the permutation

$$\pi = (1,4,2)(3,5) = (1,4)(4,2)(3,5)$$

is the permutation sending $1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 5, 4 \mapsto 2, 5 \mapsto 3$. We can also write this permutation using the notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}.$$

Remark 1.1.0.2. We will often work with the symmetric group Σ_{n+1}, which by definition is the group of permutations of the elements $\{1, 2, \ldots, n+1\}$. In this thesis, we will also use the notation Σ_{n+1} to mean permutations of $\{0, 1, \ldots, n\}$, or $\{*, 1, 2, \ldots, n\}$ for some based sets. These groups are isomorphic.

We will use the notation $\underline{n} := \{1, 2, \ldots, n\}$, and for the based version, $\underline{n}_* := \{0, 1, \ldots, n\}$.

A common presentation of the symmetric group Σ_n is the one generated by adjacent transpositions $t_i := (i, i+1)$, for $1 \leq i \leq n-1$,

$$\Sigma_n = \langle t_i : t_i^2 = 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1},\ t_i t_j = t_j t_i \text{ for } |j - i| > 1 \rangle.$$

Definition 1.1.0.3 (The cyclic group)**.** The cyclic group $C_n \subset \Sigma_n$ is the group generated by an n-cycle

$$C_n = \langle (1, 2, \ldots, n) \rangle.$$

Here we give the definition of a group action, which in our case will usually be an action of some symmetric group.

Definition 1.1.0.4 (Group action)**.** We say a group G acts on a set A, and write the action of $g \in G$ on $a \in A$ as $g \cdot a$, if

- $e \cdot a = a$,
- $g \cdot (h \cdot a) = (gh) \cdot a$,

for $g, h \in G$, where $e \in G$ is the identity group element.

We write $g \cdot A$ or gA for the action of the element g on the whole set A.

$$gA := \{g \cdot a | a \in A\}.$$

The symmetric group has an action on sets by permuting the elements of the set. The symmetric group Σ_n acts on the set $\{1, 2, \ldots, n\}$ in the obvious way by permuting elements. It acts on indexed sets $A = \{a_i | i \in \underline{n}\}$ by permuting indices.

Let A be a set with $|A| = n$. One may also define Σ_A as the symmetric group that permutes elements of A (not necessarily indexed). Then we have a group isomorphism $\Sigma_A \cong \Sigma_n$ given by a choice of bijection between A and $\underline{n}$.

We will use the notation for the standard action of Σ_n on the indexed set $A = \{a_1, a_2, \ldots a_n\}$

$$\pi A := \{a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}\}.$$

For the action of an element $\sigma \in \Sigma_Y$ on a general set $X \subset Y$, we have

$$\sigma X := \{\sigma x | x \in X\}.$$

Notice that for an unordered set A, $\sigma A = A$, for all $\sigma \in \Sigma_A$, however for an ordered set $B = (b_1, b_2, \ldots, b_n)$, the set $\tau B = (b_{\tau(1)}, \ldots, b_{\tau(n)})$ is not necessarily equal to B.

Remark 1.1.0.5. So far we have defined left group action, where the elements act on the left of a set. When we consider operads in this thesis, we will instead use right actions of the symmetric group. In this case, we have the right action of $\pi \in \Sigma_n$ on $A = \{a_1, a_2, \ldots a_n\}$ given by

$$A\pi := \{a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \ldots, a_{\pi^{-1}(n)}\}.$$

If a group G acts on a set A, we may extend linearly to get a group action on the space of linear combinations $\mathbb{Z}[A]$.

1.2 Category theory

We will rely on categorical language throughout this thesis, so we include a few key examples and properties here that are used in later chapters. The results quoted in this section can be found in a number of places, such as [Lei14], [Rie16] and [Lan98].

Most of the categories that we consider in this thesis will be symmetric monoidal categories. We will assume that any categories we use are concrete, so that we can write elements. We also assume the axiom of choice.

Definition 1.2.0.1 (Symmetric monoidal category, [Rie16, E.2])**.** A category $(\mathcal{C}, \otimes, 1)$ is monoidal if it has a product operation $\otimes$ that satisfies associativity and unit axioms. It is a symmetric monoidal category if the product is symmetric, that is, if

$$c \otimes d \cong d \otimes c,$$

for objects $c, d \in \mathcal{C}$.

We will use the notation t, for the map in a symmetric monoidal category that swaps the order of objects

$$t : c \otimes d \mapsto d \otimes c, \tag{1.1}$$

which is of course an isomorphism.

Example 1.2.0.2. The following are some examples of categories that are of interest to us or referred to in later chapters.

- Set is the category of sets and set functions.
- Σ_* is the category with objects the based sets $\underline{n}_*$ and morphisms basepoint preserving permutations $\sigma \in \Sigma_n$.
- $\mathcal{B}ij_*$ is the category of finite pointed sets and basepoint-preserving bijections.
- Σ is the category with objects the based sets $\underline{n}_*$ and morphisms the permutations $\sigma \in \Sigma_{n+1}$. Note that morphisms don't necessarily preserve the basepoint.
- $\mathcal{B}ij$ is the category of pointed finite sets and bijections which aren't required to preserve the basepoint.
- Any poset may be viewed as a category by taking the elements of the poset as the objects. Then, if $p \leq q$ in the poset, there exists a morphism $p \to q$ in the category.
- Top_* is the category of based topological spaces and basepoint preserving continuous maps. This category is symmetric monoidal with the smash product $\wedge$ as the monoidal product.
- Top is the category of topological spaces and continuous maps. This category is symmetric monoidal with the smash product $\wedge$ as the monoidal product.

- Mod_k is the category of k-modules and module homomorphisms.

- $\mathcal{SW}$ is the Spanier-Whitehead category of finite spectra, which we define in Chapter 6.

Throughout the later chapters of this thesis, we will work mainly with the categories Σ_*, $\mathcal{B}ij_*$, Σ and $\mathcal{B}ij$, and rely on the following results about skeletons. Note that the unintuitive definitions of the categories $\mathcal{B}ij$ and Σ is due to the fact we will need distinguished basepoints when defining cyclic operads in Chapter 3, but also require the ability to permute these.

Definition 1.2.0.3 (Skeleton of a category, [Rie16])**.** Let $\mathcal{C}$ be a category. Then $\mathcal{K} \subset \mathcal{C}$ is called the *skeleton* category of $\mathcal{C}$ if every object $c \in \mathcal{C}$ is isomorphic to exactly one object in $\mathcal{K}$. For any two categories $\mathcal{K}$ and $\mathcal{K}'$ that are skeletons of a category $\mathcal{C}$, there is an isomorphism $\mathcal{K} \cong \mathcal{K}'$.

A category $\mathcal{C}$ is *skeletal* if it has only one element in each isomorphism class. In which case, $\mathcal{C} = skel(\mathcal{C})$.

Proposition 1.2.0.4 ([Rie16])**.** *A category $\mathcal{C}$ is equivalent to its skeleton $\mathcal{K} \subseteq \mathcal{C}$.*

Proof (sketch). We have the inclusion $F : \mathcal{K} \hookrightarrow \mathcal{C}$. The functor F is fully faithful and essentially surjective. Therefore, assuming the axiom of choice, it gives an equivalence of categories. □

For each object $c \in \mathcal{C}$, we will denote by $\iota_c : c \mapsto \overline{c}$ the isomorphism to the class representative $\overline{c}$ in a skeleton $\mathcal{K}$ of $\mathcal{C}$.

Example 1.2.0.5 ($\mathcal{B}ij_* \simeq \Sigma_*$)**.** The category Σ_* is the skeleton of Bij_*. Let $(X, x_0), (Y, y_0) \in \mathcal{B}ij_*$ and $\pi : (X, x_0) \mapsto (Y, y_0) \in \mathcal{B}ij_*$ a bijection. Let R and E be the restriction and extension functors as used by Lukács [Luk13]. Note that E is the inclusion as F is in proposition 1.2.0.4. Then $RE = id$ and we have the following diagram of equivalence maps.

$$\mathcal{B}ij_* \underset{E}{\overset{R}{\rightleftarrows}} \Sigma_* \qquad \begin{array}{ccc} (X,x_0) & \xmapsto{\iota_{(X,x_0)}} & \underline{n}_* \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \iota_{(Y,y_0)}\pi\iota^{-1}_{(X,x_0)}} \\ (Y,y_0) & \xmapsto[\iota_{(Y,y_0)}]{} & \underline{n}_* \end{array}$$

Example 1.2.0.6 ($\mathcal{B}ij \simeq \Sigma$)**.** The category Σ is the skeleton of Bij. Let $X, Y \in \mathcal{B}ij$, and $\pi : X \mapsto Y \in \mathcal{B}ij$ be a bijection. Let R and E again be the restriction and extension functors. We will use the same names R and E for the restriction and extension functors in both the based and unbased setting, as it will be clear which setting we are working in. Then we have the diagram below.

$$\mathcal{B}ij \underset{E}{\overset{R}{\rightleftarrows}} \Sigma \qquad \begin{array}{ccc} X & \xmapsto{\iota_X} & \underline{n} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \iota_Y\pi\iota^{-1}_X} \\ Y & \xmapsto[\iota_Y]{} & \underline{n} \end{array}$$

In fact, all the above categories are equivalent. The diagram below shows the relationship, and in particular we have the horizontal forgetful functors U.

$$\begin{array}{ccc} \mathcal{B}ij_* & \xrightarrow{U} & \mathcal{B}ij \\ E\Big\uparrow\Big\downarrow R & & E\Big\uparrow\Big\downarrow R \\ \Sigma_* & \xrightarrow[U]{} & \Sigma \end{array}$$

The functor U is an equivalence because it is identity on objects, and on morphisms we forget the basepoint non-preserving bijections.

Definition 1.2.0.7 (Functor category)**.** For categories $\mathcal{C}, \mathcal{D}$ with $\mathcal{D}$ small, the category $Fun(\mathcal{D}, \mathcal{C})$ has as objects functors $\mathcal{D} \xrightarrow{F} \mathcal{C}$, and as morphisms, natural transformations.

Proposition 1.2.0.8. *The assignment $\mathcal{D} \mapsto Fun(\mathcal{D}, \mathcal{C})$ preserves equivalences.*

Proof. For equivalent categories $\mathcal{D}$ and $\mathcal{D}'$, we have inverse equivalences F and G,

$$\mathcal{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}'.$$

This induces an equivalence on functor categories as follows

$$Fun(\mathcal{D}, \mathcal{C}) \underset{F^*}{\overset{G^*}{\rightleftarrows}} Fun(\mathcal{D}', \mathcal{C})$$

$$H \longmapsto H \circ G$$
$$K \circ F \longleftarrow K$$

Then we have

$$F^*G^*(H) = H \circ G \circ F \cong H$$

for any $H \in Fun(\mathcal{D}, \mathcal{C})$, since $GF \cong id_{\mathcal{D}}$. Therefore, $F^*G^* \cong id_{Fun(\mathcal{D},\mathcal{C})}$. Similarly, we have $G^*F^* \cong id_{Fun(\mathcal{D}',\mathcal{C})}$, and therefore F^* and G^* are inverse equivalences. □

Note that the above result could also be proved by the fact that G^* is a 2-functor, which preserves equivalences.

Definition 1.2.0.9 (Opposite category)**.** For a category $\mathcal{D}$, we define the opposite category $\mathcal{D}^{\text{op}}$ with the same objects as $\mathcal{D}$, and morphisms $f : d \to c$ for each morphism $f : c \to d$ in $\mathcal{D}$.

Later we will need the fact that taking the opposite also preserves equivalences. That is, for equivalent categories $\mathcal{D} \simeq \mathcal{D}'$,we have $\mathcal{D}^{op} \simeq \mathcal{D}'^{op}$.

1.3 Sets

There are some properties of sets, and structures we can build from finite sets that will be important in our discussion of partitions and trees.

Definition 1.3.0.1 (Disjoint union)**.** The disjoint union is the coproduct in the category of sets. Below is a model for disjoint union of finite sets. Let $A = \{a_1, a_2, \dots, a_m\}$, $B = \{b_1, b_2, \dots, b_n\}$ be finite sets. The disjoint union is defined by first indexing the sets A, B:

$$A' = \{(a_1, 0), (a_2, 0), \dots, (a_m, 0)\},\ B' = \{(b_1, 1), (b_2, 1), \dots, (b_n, 1)\}.$$

Then

$$A \sqcup B := A' \cup B' = \{(a_1, 0), (a_2, 0), \dots, (a_m, 0), (b_1, 1), (b_2, 1), \dots, (b_n, 1)\}.$$

In general for finite sets $\{A_i | i \in I\}$ for some indexing set I, we denote the indexed sets by $A_i' = \{(a, i) | a \in A_i\}$. Then we have the arbitrary disjoint union

$$\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} A_i'.$$

Since the disjoint union is the categorical coproduct, it is associative up to canonical isomorphism.

Remark 1.3.0.2. We will use notation $\{a_1, \dots, a_n\}$ for finite (unordered) sets, and $(a_1, \dots, a_n)$ for finite ordered sets.

1.4 General representation theory

In this section we will give some relevant background material on representations and in particularly symmetric group representations. We mainly use the language and descriptions of Sagan in [Sag01], and another good resource for symmetric group representation theory is [Jam87].

Representation theory allows one to model structures such as groups and algebras using matrices or linear transformations.

Definition 1.4.0.1 (Group representation)**.** A matrix representation of a group G is a group homomorphism $\Phi : G \mapsto GL_d$, where GL_d is the general linear group (over $\mathbb{C}$) and we call d the dimension or degree of the representation, $deg(\Phi)$.

In terms of modules, a vector space V is called a G-module if there is a group homomorphism $\rho : G \mapsto GL(V)$.

In general, one can define representations over any base field, but for our purposes we will always work over $\mathbb{C}$ unless stated otherwise.

We may also describe group representations using categorical language. A group representation is a functor $F : G \to V$ from the single object category formed from the group G, to the category of vector spaces over $\mathbb{C}$.

1.4.1 Reducibility and characters

One of the most important theorems in representation theory is Maschke's theorem. This says that representations can be decomposed uniquely into a direct sum of irreducible representations.

Theorem 1.4.1.1 (Maschke's theorem)**.** *For a finite group G, and nonzero G-module V,*

$$V = W^{(1)} \oplus \cdots \oplus W^{(k)},$$

for irreducible G-submodules $W^{(i)}$ of G.

Characters contain a lot of information about a representation and can be used to check reducibility.

Definition 1.4.1.2 (Characters)**.** For a matrix representation $X(g)$ with $g \in G$, the character of X is defined as

$$\chi(g) := trX(g),$$

the trace of the matrix.

The characters of all the irreducible representations of a group are often presented in a character table with rows labelled by the irreducible representations and columns labelled by representatives of equivalence classes. Then, if we know the characters for a particular reducible representation of a group, we only need to compare these with the character table for the group in order to determine the unique decomposition into irreducible representations.

1.4.2 Induced and restricted representations

When we consider actions of subgroups $H \subset G$ as restrictions of the action of the bigger group G, then the associated representations will be restricted representations. Conversely, when an action of H is extended to an action of the bigger group G, we get induced representations.

Definition 1.4.2.1 (Restricted representation [Sag01, Definition 1.12.1])**.** For a group G, subgroup $H \subset G$, and a matrix representation X of G, the restriction $X \downarrow_H^G$ or $Res_H^G X$ of X to G is defined by the following.

$$X \downarrow_H^G (h) := X(h), \quad \text{for all } h \in H.$$

Definition 1.4.2.2 (Induced representation)**.** For a group G, subgroup $H \subset G$, and a matrix representation Y of G, the induced representation $Y \uparrow_H^G$ or $Ind_H^G X$ of Y to G is defined by the following.

$$Y \uparrow_H^G (g) := (Y(t_i^{-1} g t_j)),$$

where $(Y(t_i^{-1} g t_j))$ is a block matrix, and $t_1, t_2, \ldots, t_k$ such that

$$G = t_1 H \sqcup t_2 H \sqcup \cdots \sqcup t_k H.$$

We call $t_1, \ldots, t_k$ a transversal for the left cosets of H in G.

1.5 Representation theory of the symmetric group

Since this thesis is largely an exploration of actions of the symmetric group, we will specify most of the background representation theory to this case. There are a number of objects, diagrams and properties specific to symmetric group representations which make it significantly easier to study than the representations of general groups.

Example 1.5.0.1 (Examples)**.** Below are some important examples of Σ_n representations.

- The (left/right) *regular representation* of a group G is associated to the (left/right) action of G on itself by multiplication.
- The *trivial representation* sends every $g \in G$ to the identity matrix.
- For $\pi \in \Sigma_n$, the representation $X(\pi) = sgn(\pi)$ is called the *sign representation.*
- The *defining representation* of Σ_n is the one where each $X(\pi)$ is the permutation matrix.

For representation theory of the symmetric group, integer partitions play a large part. We will introduce notation for these.

Definition 1.5.0.2 (Integer partition of $n \in \mathbb{N}$)**.** An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n \in \mathbb{N}$ is a collection of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$, such that $\sum_{i=1}^{k} \lambda_i = n$. One may also use the notation $\lambda \vdash n$ if λ is a partition of n.

We have orderings on both set partitions and integer partitions. We will talk about the ordering on set partitions later. An ordering on integer partitions that is important is the dominance ordering.

Definition 1.5.0.3 (Dominance)**.** We say a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *dominates* a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, written $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{r} \lambda_i \geq \sum_{i=1}^{r} \mu_i, \qquad \text{for all } r \leq \min(k, l).$$

1.5.1 Young diagrams

We illustrate Σ_n representations using diagrams called *Young tableaux*. These are indexed by partitions $\lambda := (\lambda_1, \ldots, \lambda_r)$ of n. A Young tabloid of shape λ is an array of r rows such that row i contains λ_i elements, the order of which doesn't matter. For example, the tabloid below has shape $\lambda = (4, 3, 2, 2)$ and content $\mu = \{1, 2, \ldots, 11\}$.

4	6	1	7
2	5	10	
8	3		
9	11		

The irreducible modules are called *Specht modules.* These are cyclic modules that are generated by Young tableaux.

Definition 1.5.1.1 (Specht modules [Sag01, Theorem 2.5.2])**.** The Specht module S^λ of shape λ, where $\lambda \vdash n$ is a partition of n and e_t is a polytabloid, is

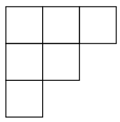
$$S^\lambda := span\{e_t | t \text{ a } \lambda\text{-tableau}\},$$

For the symmetric group, the Specht modules are the building blocks of any representation, and we often write representations as direct sums of Specht modules.

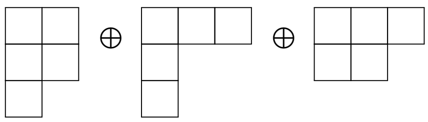
We can use Young tableaux to show restricted and induced representations via the *branching rule.*

Proposition 1.5.1.2 (Branching rule)**.** *For a partition $\lambda \vdash n$, the restriction $S^\lambda\downarrow^{\Sigma_n}_{\Sigma_{n-1}}$ of the representation S^λ to Σ_{n-1} is given by the sum of representations associated with the diagrams resulting from removing a cell in the allowed way. Similarly, the induced representation $S^\lambda\uparrow^{\Sigma_{n+1}}_{\Sigma_n}$ is the sum of Σ_{n+1} representations resulting from adding a cell to the Young diagram.*

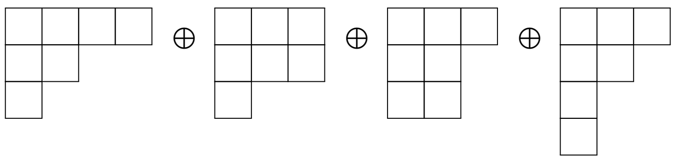
For example, the representation $S^{(3,2,1)}$ has the associated diagram



with restriction to Σ_5



and induction to Σ_7 as below.



The following rules and definitions are useful for doing character calculations and determining representations.

Definition 1.5.1.3 (Permutation module)**.** The permutation module M^μ for $\mu \vdash n$ is a Σ_n-module spanned by all possible μ-tabloids.

The dimension of a permutation module is given by

$$\dim M^\mu = \frac{n!}{\mu!}.$$

The permutation modules M^μ can be decomposed into irreducible terms given by the Specht modules S^λ.

Proposition 1.5.1.4 (Young's rule)**.**

$$M^\mu \cong \bigoplus_\lambda \kappa_{\lambda\mu} S^\lambda,$$

where the coefficients $\kappa_{\lambda\mu}$ are the Kostka numbers.

Definition 1.5.1.5 (Kostka numbers)**.** The Kostka number $\kappa_{\lambda\mu}$ for permutations $\lambda \vdash n, \mu \vdash n$ is given by

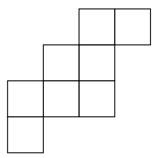
$$\kappa_{\lambda\mu} := \#\{\text{semi-standard } \lambda - \text{tableaux of type } \mu\}.$$

There is a convenient formula for the number f^λ of standard λ-tableaux.

Proposition 1.5.1.6 (Hook formula)**.** *The number of standard λ tableaux is given by*

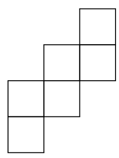
$$f^\lambda = n!\left|\frac{1}{(\lambda_i - i + j)!}\right|.$$

Definition 1.5.1.7 (Skew diagram)**.** A skew tableau or diagram is a Young diagram of the type pictured below.



We say the tableau has shape λ/μ for partitions $\lambda = (\lambda_1, \lambda_2 \dots)$, $\mu = (\mu_1, \mu_2, \dots)$ such that $\mu_i \le \lambda_i$ for all i. Then the skew diagram is the 'difference' of the diagrams of λ and μ.

Definition 1.5.1.8 (Border strip)**.** A border strip is a connected subset of a Young tableau, that does not contain a 2×2 square. For example the following is a border strip in the above skew diagram:



There is a formula called the Murnaghan-Nakayama rule that allows us to calculate the irreducible characters using only information from the Young tableaux. This means we can construct the character tables for any (small enough to calculate with) value of n. In particular, it takes information from the border strips of a given Young diagram.

Proposition 1.5.1.9 (Murnaghan-Nakayama rule, [KW20, Theorem 1.1])**.** *For an n-cycle $\tau \in \Sigma_{m+n}$, and permutation π of the remaining m numbers, we have*

$$\chi^\lambda(\pi\tau) = \sum (-1)^{ht(\lambda/\mu)} \chi^\mu(\pi),$$

summing over $\mu \subset \lambda$ with $|\mu| = m$, such that λ/μ is a border strip, and $ht(\lambda/\mu)$ is one less than the number of non-empty rows of the skew diagram of shape λ/μ.

Chapter 2

Operads

Operads are a structure first define by May in [May72], with the purpose of studying loop spaces. They have applications in homotopy theory, algebra, category theory and many other areas of mathematics and physics.

In this chapter, we will explore a number of different ways to define operads. In particular, our focus will be on the skeletal and non-skeletal definitions of the form 'symmetric sequence (or collection) with partial composition'.

The main aim of the chapter is to discuss in detail the proof by Lukács that the skeletal and non-skeletal definitions are equivalent. We give this result in Theorem 2.4.2.4. In later chapters, we will use this method to prove the equivalences of the skeletal and non-skeletal definitions of cyclic operads, cooperads and cyclic cooperads. We provide a clear set up with all key functors and necessary constructions defined. In particular, we will discuss the disjoint union in Set_*, and the fact that this and related operations are not strictly associative.

We will introduce clear notation in order to carefully distinguish the based and unbased situations, where the unbased setting will be of particular use to us in the next chapter, and also make clear throughout which categories objects belong to. Where in the literature many of the axiom checks are left to the reader, we will include a number of these checks in full detail. In this way, we lay the foundations for the corresponding result for cyclic operads in Chapter 3.

Finally, we will look at some examples and properties. In particular, the associative operad will be a running example throughout the thesis.

2.1 Background and definitions

In this chapter we will work with the categories Σ_* and $\mathcal{B}ij_*$. Recall that Σ_* is the category of pointed sets $\underline{n}_* := \{0, 1, 2, \ldots, n\}$ and permutations in Σ_n for $n \in \mathbb{N}$, and $\mathcal{B}ij_*$ is the category of finite pointed sets and basepoint preserving bijections. These are equivalent as described in example 1.2.0.5.

We will need a definition of a kind of deleted disjoint union in order to set up the non-skeletal definition of an operad. The below definition is a model for this construction in Set_*, the category of pointed sets. Unlike the usual disjoint union in *Set* (Definition 1.3.0.1), the deleted disjoint union is not the coproduct in Set_*, since the point associated with the union is not the basepoint of the set.

Definition 2.1.0.1 (Deleted disjoint union)**.** Let $(X, x_0), (Y, y_0)$ be pointed finite sets. Then for $x \in X \setminus \{x_0\}$, define the deleted disjoint union $(X \sqcup_x Y, (x_0, 0))$ at x by

$$X \sqcup_x Y := X \sqcup Y \setminus \{(x, 0), (y_0, 1)\}.$$

Proposition 2.1.0.2. *The operation $\sqcup_x$ satisfies associativity up to isomorphism:*

$$(X \sqcup_x Y) \sqcup_{x'} Z \cong (X \sqcup_{x'} Z) \sqcup_x Y \text{ for } x' \in X$$
$$(X \sqcup_x Y) \sqcup_y Z \cong X \sqcup_x (Y \sqcup_y Z) \text{ for } y \in Y,$$

where $(X, x_0), (Y, y_0)$ and (Z, z_0) are finite pointed sets. The first case is given by composing in X followed by X again, and the second case is composing in X followed by Y.

Proof. Everything here is up to isomorphism. For example, by our definition of disjoint union, $(X \sqcup Y) \sqcup Z$ technically has terms that look like $((x, 0), 0)$. However, we can simply choose canonical bijections to the expressions in the proof.

Define the indexed sets $X' = \{(x, 0)|x \in X\}$, $Y' = \{(y, 1)|y \in Y\}$, $Z' = \{(z, 2)|z \in Z\}$. Then we have the canonical isomorphisms

$$(X \sqcup_x Y) \sqcup_{x'} Z \cong (X \sqcup_{x'} Z) \sqcup_x Y \cong X' \cup Y' \cup Z' \setminus \{(x, 0), (y_0, 1), (x', 0), (z_0, 2)\}.$$

Similarly, in the other case, we have

$$(X \sqcup_x Y) \sqcup_y Z \cong X \sqcup_x (Y \sqcup_y Z) \cong X' \cup Y' \cup Z' \setminus \{(x, 0), (y_0, 1), (y, 1), (z_0, 2)\}.$$

□

In what follows, we follow standard conventions and will treat these canonical isomorphisms as identities. We will also abbreviate notation, so, for example, we write $x' \in X \sqcup_x Y$ rather than $(x', 0) \in X \sqcup_x Y$ and so on.

2.1.1 Renumbering map

The following renumbering map plays an essential part in the skeletal definition of an operad, although it often isn't mentioned at all in the literature. We will defer most of the discussion of this map and how it differentiates the skeletal case from the non-skeletal case to Section 2.4. However, we will define it here, as it appears first in the composition of permutations that we define below, and these are a large part of the skeletal definition.

Definition 2.1.1.1 (Renumbering map φ_i)**.** The renumbering map $\varphi : \Sigma_* \to \mathcal{B}ij_*$ is given by

$$\varphi_i(k) := \begin{cases} (k, 0) \in \underline{m}_* \times \{0\} & \text{if } k < i, \\ (k - n + 1, 0) \in \underline{m}_* \times \{0\} & \text{if } k > i + n - 1, \\ (k - i + 1, 1) \in \underline{n}_* \times \{1\} & \text{if } i \leq k \leq i + n - 1. \end{cases}$$

The indexing elements 0 and 1 are a technicality from the deleted disjoint union, and in practice we will not write these.

2.1.2 Composition of maps

When we consider equivariance of the composition operation, we will need composition of permutations or bijections in the skeletal and non-skeletal setting respectively. Throughout the rest of this thesis, we will use the notation $\bullet$ for any type of skeletal composition that involves the renumbering map, and $\circ$ for non-skeletal composition that doesn't involve the renumbering map.

First we will define the composition of bijections in $\mathcal{B}ij_*$, which is itself a bijection in $\mathcal{B}ij_*$.

Definition 2.1.2.1 (Composition of set bijections [Luk13]). Let $\rho : (X, x_0) \to (X', x'_0)$ and $\pi : (Y, y_0) \to (Y', y'_0) \in \mathcal{B}ij_*$ be bijections, and $x \in X, x \neq x_0$. Then we define the composition

$$\rho \circ_x \pi : (X \sqcup_x Y, x_0) \to (X' \sqcup_{\rho(x)} Y', x'_0)$$

to be the bijection that restricts as

$$\rho \circ_x \pi|_{X \setminus \{x\}} := \rho|_{X \setminus \{x\}}$$
$$\rho \circ_x \pi|_{Y \setminus \{y_0\}} := \pi|_{Y \setminus \{y_0\}}.$$

The following composition of permutations is one we don't see directly in the definition of a skeletal operad, however it underlies in the structure, and it is necessary when passing to the non-skeletal case. In fact, it is simply a special case of the non-skeletal composition of bijections defined above, where the finite sets are the sets $\underline{m}_*$, and the bijections those that map $\underline{m}_* \to \underline{m}_*$.

Definition 2.1.2.2 (Composition of permutations in Σ_* viewed as bijections in $\mathcal{B}ij_*$). Let $\sigma \in \Sigma_m, \tau \in \Sigma_n$ be permutations, and $i \in \underline{m}$. Then we define the composition $\sigma \circ_i \tau \in \mathcal{B}ij_*$

$$\sigma \circ_i \tau : \underline{m}_* \sqcup_i \underline{n}_* \to \underline{m}_* \sqcup_i \underline{n}_*$$

to be the permutation that restricts as

$$\sigma \circ_i \tau|_{\underline{m}_* \setminus \{i\}} := \sigma|_{\underline{m}_* \setminus \{i\}}$$
$$\sigma \circ_i \tau|_{\underline{n}} := \tau|_{\underline{n}}.$$

Finally, we will define the composition of permutations in Σ_*, which is itself a permutation of a larger set.

Definition 2.1.2.3 (Composition of permutations in Σ_*). Let $\sigma \in \Sigma_m, \tau \in \Sigma_n$ be permutations, and $i \in \underline{m}$. Then we define the composition $\sigma \bullet_i \tau \in \Sigma_{m+n-1}$

$$\sigma \bullet_i \tau : \underline{m+n-1}_* \to \underline{m+n-1}_*$$

to be the permutation such that the following diagram commutes.

$$
\begin{array}{ccc}
\underline{m}_* \sqcup_i \underline{n}_* & \xrightarrow{\varphi_i^{-1}} & \underline{m+n-1}_* \\
\big\downarrow {\scriptstyle \sigma \circ_i \tau} & & \big\downarrow {\scriptstyle \sigma \bullet_i \tau} \\
\underline{m}_* \sqcup_i \underline{n}_* & \xrightarrow[\varphi_{\sigma(i)}^{-1}]{} & \underline{m+n-1}_*
\end{array}
$$

We will now give the definition of a symmetric sequence. Throughout this chapter, let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

Definition 2.1.2.4 (Symmetric sequence)**.** A symmetric sequence in a symmetric monoidal category $\mathcal{C}$, for $X \in \mathcal{C}$, is a sequence $\{X(n)\}_{n \in \mathbb{N}_{>0}}$ with a right action of Σ_n on each $X(n)$. This is equivalently a functor $\Sigma^{op} \to \mathcal{C}$.

Remark 2.1.2.5. A Σ-module in $\mathcal{C}$ is functor $\Sigma^{op} \to \mathcal{C}$. This is also sometimes referred to in this context as a collection, and elsewhere in the literature as a species. Note that $\Sigma \simeq \Sigma^{op}$, and in the literature different authors take functors $\Sigma \to \mathcal{C}$.

2.2 Skeletal operad definitions

The standard definitions of operads given by May in [May72] is skeletal. That is, they define an operad and its structure in terms of the based sets $\underline{n}_*$, for natural numbers n, together with permutations in the associated symmetric groups Σ_n.

Definition 2.2.0.1 (Skeletal operad)**.** An operad is a symmetric sequence P in $\mathcal{C}$ along with a unit $\eta : \mathbf{1} \to P(1)$ and partial composition operations

$$\bullet_i : P(m) \otimes P(n) \to P(n+m-1), \quad i \in \underline{n}$$

that satisfy equivariance, associativity and unit axioms. These axioms can be presented in the form of commutative diagrams.

- (Equivariance) Let $\sigma \in \Sigma_m, \tau \in \Sigma_n$, $i \in \underline{m}_*$, then the following diagram commutes.

$$
\begin{array}{ccc}
P(m) \otimes P(n) & \xrightarrow{\bullet_{\sigma(i)}} & P(m+n-1) \\
\big\downarrow {\scriptstyle \sigma \otimes \tau} & & \big\downarrow {\scriptstyle \sigma \bullet_i \tau} \\
P(m) \otimes P(n) & \xrightarrow[\bullet_i]{} & P(m+n-1)
\end{array}
$$

- (Associativity) Let $1 \leq i \leq l$ and $i \leq j \leq i+m-1$, that is, we first compose $q \in P(m)$ with $p \in P(l)$ followed by $r \in P(n)$ with $q \in P(m)$. Then we get the diagram below.

$$
\begin{array}{ccc}
P(l) \otimes P(m) \otimes P(n) & \xrightarrow{\bullet_i \otimes id} & P(l+m-1) \otimes P(n) \\
\big\downarrow {\scriptstyle id \otimes \bullet_j} & & \big\downarrow {\scriptstyle \bullet_{j+i-1}} \\
P(l) \otimes P(m+n-1) & \xrightarrow[\bullet_i]{} & P(l+m+n-2)
\end{array}
$$

Let $1 \le i \le l$ and $j < i$. That is, we compose $q \in P(m)$ with $p \in P(l)$ followed by $r \in P(n)$ with $p \in P(l)$. Let t be the map that swaps the order in the tensor product as in (1.1). Then we have the following commutative diagram.

$$\begin{array}{ccc}
P(l) \otimes P(m) \otimes P(n) & \xrightarrow{\bullet_i \otimes id} & P(l+m-1) \otimes P(n) \\
\downarrow{\scriptstyle id \otimes t} & & \\
P(l) \otimes P(n) \otimes P(m) & & \downarrow{\scriptstyle \bullet_j} \\
\downarrow{\scriptstyle \bullet_j \otimes id} & & \\
P(l+n-1) \otimes P(m) & \xrightarrow[\bullet_i]{} & P(l+m+n-2)
\end{array}$$

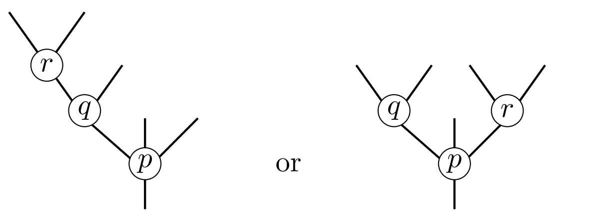
If $j > i$, we have the diagram below.

$$\begin{array}{ccc}
P(l) \otimes P(m) \otimes P(n) & \xrightarrow{\bullet_i \otimes id} & P(l+m-1) \otimes P(n) \\
\downarrow{\scriptstyle id \otimes t} & & \\
P(l) \otimes P(n) \otimes P(m) & & \downarrow{\scriptstyle \bullet_{j+m-1}} \\
\downarrow{\scriptstyle \bullet_j \otimes id} & & \\
P(l+n-1) \otimes P(m) & \xrightarrow[\bullet_i]{} & P(l+m+n-2)
\end{array}$$

- (Unit) Let $\mathbf{1} \in \mathcal{C}$ be the unit object of the underlying category. Then there is a map $\eta : \mathbf{1} \to P(1)$ that interacts with the composition operation such that the following diagrams commute for all $i \in \underline{n}$.

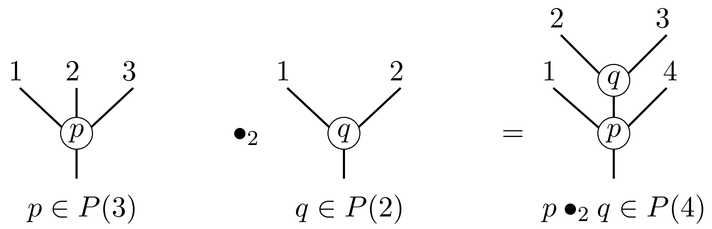
$$\begin{array}{ccc}
P(n) \otimes \mathbf{1} & \xrightarrow{id \otimes \eta} & P(n) \otimes P(1) \\
& \searrow & \downarrow{\scriptstyle \bullet_1} \\
& & P(n)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1} \otimes P(n) & \xrightarrow{\eta \otimes id} & P(1) \otimes P(n) \\
& \searrow & \downarrow{\scriptstyle \bullet_i} \\
& & P(n)
\end{array}$$

Remark 2.2.0.2. We see that there are three commutative diagrams for the associativity axiom. This is because when we do two consecutive partial compositions, there are two cases which depend on which of the first two operations the third is composed with. The second case splits into two cases in the skeletal setting, depending on whether the second composition is to the left or right of the first. This shifts the indexing because of the renumbering map. Using an example with tree diagrams, the difference between the main two cases is depicted as follows.



$(p \bullet_1 q) \bullet_1 r \in P(5)$ $\qquad$ $(p \bullet_1 q) \bullet_4 r \in P(5)$

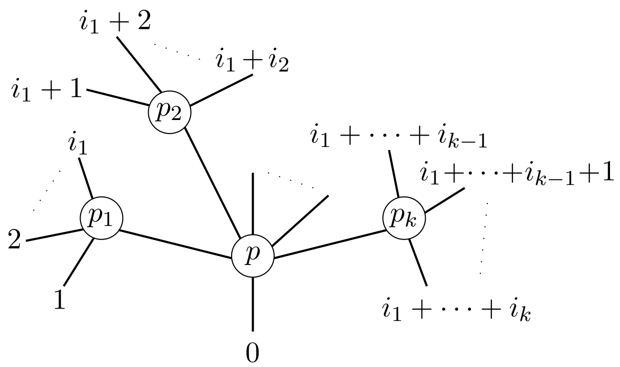
It is convenient to depict n-ary operations as rooted n-trees where the labelled leaves are the inputs, the vertex is the operation and the root represents the output. Then composition of operations is analogous to grafting trees.



Remark 2.2.0.3. Rather than using the partial composition operations $\bullet_i$, the classical skeletal definition of an operad gives a sequence of sets with symmetric group actions and total composition operations

$$P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k) \to P(i_1 + \cdots + i_k),$$

where a number k of operations are composed with an operation in $P(k)$, one in each input position. With tree diagrams this looks like the following.



The formulation with partial composition operations $\bullet_i$ is an equivalent, and somewhat easier to work with, definition. We can obtain the general case from the partial case by successive $\bullet_i$ operations:

$$(\ldots(((p \bullet_1 p_1) \bullet_{i_1+1} p_2) \bullet_{i_1+i_2+1} p_3) \cdots \bullet_{i_1+\cdots+i_{k-1}+1} p_k).$$

In the other direction, one obtains partial compositions from general compositions by taking all but the i^{th} inserted operation to be the identity. That is,

$$p \bullet_i q = p(1, \ldots, 1, q, 1, \ldots, 1).$$

This is discussed in more detail by Fresse in [Fre17].

Remark 2.2.0.4. An operad P can also be defined as a monoid in the category of symmetric sequences as described in [Chi05, Proposition 2.9].

There is a natural notion of a map of operads, so we have a category.

Definition 2.2.0.5 (Category of skeletal operads)**.** We denote by $\mathcal{O}p_{\Sigma_*}$ the category with objects skeletal operads, and morphisms the maps $\theta : P \to Q$ for $P, Q \in \mathcal{O}p_{\Sigma_*}$, where $\theta_n : P(n) \to Q(n)$. The maps θ_n are compatible with the Σ_n action, partial composition operations and unit. These conditions are encoded by the following commutative diagrams.

- (Compatibility with Σ_n action)

$$\begin{array}{ccc} P(n) & \xrightarrow{\theta_n} & Q(n) \\ \sigma \downarrow & & \downarrow \sigma \\ P(n) & \xrightarrow[\theta_n]{} & Q(n), \end{array}$$

  where $\sigma : \underline{n} \to \underline{n}$ is a permutation.

- (Compatibility with $\bullet_i$)

$$\begin{array}{ccc} P(m) \otimes P(n) & \xrightarrow{\theta_m \otimes \theta_n} & Q(m) \otimes Q(n) \\ \bullet_i \downarrow & & \downarrow \bullet_i \\ P(m+n-1) & \xrightarrow[\theta_{m+n-1}]{} & Q(m+n-1) \end{array}$$

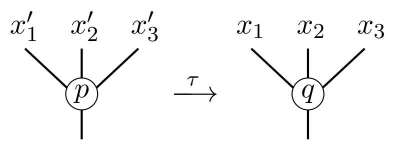
- (Compatibility with unit)

$$\begin{array}{ccc} \mathbf{1} & =\!=\!= & \mathbf{1} \\ \eta \downarrow & & \downarrow \eta \\ P(1) & \xrightarrow[\theta_1]{} & Q(1), \end{array}$$

  where η is as in Definition 2.2.0.1

2.3 Non-skeletal operad definitions

The diagram below shows the labelling of a non-skeletal operad by elements of a finite set $X = \{x_1, x_2, x_3\}$ and applying a bijection $\tau : X \to X'$, where $X' = \{x'_1, x'_2, x'_3\}$.



$p \in P(X', x'_0)$ $\qquad$ $q \in P(X, x_0)$

We will give a non-skeletal definition of the same type as the skeletal definition above.

As in the skeletal case, we have a $\mathcal{B}ij_*$-collection given by a functor $\mathcal{B}ij_*^{op} \to \mathcal{C}$.

Definition 2.3.0.1 (Non-skeletal operad)**.** An operad is a $\mathcal{B}ij_*$-collection P along with a unit $\eta : \mathbf{1} \to P(W)$ and for $x \in X \setminus \{x_0\}$, partial composition maps

$$\circ_x : P(X, x_0) \otimes P(Y, y_0) \mapsto P(X \sqcup_x Y, x_0).$$

These maps are required to satisfy equivariance, associativity and unit axioms. These are given by commutativity of the diagrams below. Let $(X, x_0), (Y, y_0), (Z, z_0) \in \mathcal{B}ij_*$ be based finite sets, and $x \in X \setminus \{x_0\}$, $y \in Y \setminus \{y_0\}$, $z \in Z \setminus \{z_0\}$.

- (Equivariance)

$$\begin{array}{ccc} P(X', x_0') \otimes P(Y', y_0') & \xrightarrow{\circ_{\sigma(x)}} & P(X' \sqcup_{\sigma(x)} Y', x_0') \\ {\scriptstyle \sigma\otimes\tau}\big\downarrow & & \big\downarrow{\scriptstyle \sigma\circ_x\tau} \\ P(X, x_0) \otimes P(Y, y_0) & \xrightarrow[\circ_x]{} & P(X \sqcup_x Y, x_0) \end{array}$$

- (Associativity) Composing first in position $x \in X$, then in $y \in Y$ gives the diagram below

$$\begin{array}{ccc} P(X, x_0) \otimes P(Y, y_0) \otimes P(Z, z_0) & \xrightarrow{\circ_x \otimes id} & P(X \sqcup_x Y, x_0) \otimes P(Z, z_0) \\ {\scriptstyle id\otimes\circ_y}\big\downarrow & & \big\downarrow{\scriptstyle \circ_y} \\ P(X, x_0) \otimes P(Y \sqcup_y Z, y_0) & \xrightarrow[\circ_x]{} & P(X \sqcup_x Y \sqcup_y Z, x_0). \end{array}$$

Composing in $x \in X$ followed by $x' \in X$ gives

$$\begin{array}{ccc} P(X, x_0) \otimes P(Y, y_0) \otimes P(Z, z_0) & \xrightarrow{\circ_x \otimes id} & P(X \sqcup_x Y, x_0) \otimes P(Z, z_0) \\ {\scriptstyle id\otimes t}\big\downarrow & & \\ P(X, x_0) \otimes P(Z, z_0) \otimes P(Y, y_0) & & \big\downarrow{\scriptstyle \circ_{x'}} \\ {\scriptstyle \circ_{x'}\otimes id}\big\downarrow & & \\ P(X \sqcup_{x'} Z, x_0) \otimes P(Y, y_0) & \xrightarrow[\circ_x]{} & P(X \sqcup_x Y \sqcup_{x'} Z, x_0) \end{array}$$

- (Unit) There exists a map $\eta : \mathbf{1} \to P(W, w_0)$ where $(W, w_0) = \{w, w_0\}$, the set with one non-basepoint element such that the following diagrams commmute for all $x \in X$.

$$\begin{array}{ccc} P(X, x_0) \otimes \mathbf{1} & \xrightarrow{id\otimes\eta} & P(X, x_0) \otimes P(W, w_0) \\ & \searrow & \big\downarrow{\scriptstyle \circ_w} \\ & & P(X, x_0) \end{array} \qquad \begin{array}{ccc} \mathbf{1} \otimes P(X, x_0) & \xrightarrow{\eta\otimes id} & P(W, w_0) \otimes P(X, x_0) \\ & \searrow & \big\downarrow{\scriptstyle \circ_x} \\ & & P(X, x_0) \end{array}$$

The collection of non-skeletal operads also form a category.

Definition 2.3.0.2 (Category of non-skeletal operads)**.** We denote by $\mathcal{O}p_{\mathcal{B}ij_*}$ the category with objects $\mathcal{B}ij_*$-operads, and morphisms the maps $\theta : P \to Q$ for $P, Q \in \mathcal{O}p_{\mathcal{B}ij_*}$, where $\theta_X : P(X, x_0) \to Q(X, x_0)$. The maps θ_X are compatible with the bijections in $\mathcal{B}ij_*$, partial composition operations and unit. These conditions are encoded by the following commutative diagrams.

- (Compatibility with bijections)

$$\begin{array}{ccc} P(X', x_0') & \xrightarrow{\theta'_X} & Q(X', x_0') \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ P(X, x_0) & \xrightarrow[\theta_X]{} & Q(X, x_0), \end{array}$$

  where $\rho : X \to X'$ is a bijection.

- (Compatibility with $\circ_x$)

$$\begin{array}{ccc} P(X, x_0) \otimes P(Y, y_0) & \xrightarrow{\theta_X \otimes \theta_Y} & Q(X, x_0) \otimes Q(Y, x_0) \\ {\scriptstyle\circ_x}\downarrow & & \downarrow{\scriptstyle\circ_x} \\ P(X \sqcup_x Y, x_0) & \xrightarrow[\theta_{X \sqcup_x Y}]{} & Q(X \sqcup_x Y, x_0). \end{array}$$

- (Compatibility with unit)

$$\begin{array}{ccc} \mathbf{1} & = & \mathbf{1} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ P(W, w_0) & \xrightarrow[\theta_W]{} & Q(W, w_0), \end{array}$$

  where η is as in Definition 2.3.0.1.

2.4 Equivalence of skeletal and non-skeletal definitions

The equivalence of the skeletal and non-skeletal definitions is proved by Lukács in [Luk13, Theorem 4.4]. We expand on the proof given by Lukács in order to provide more detail and clarify notation. In particular, this proof gives an equivalence of categories.

Remark 2.4.0.1. Note that for a permutation $\sigma : \underline{n} \to \underline{n}$ or a bijection $\tau : X \to X'$, we have the induced maps on operads $P(\sigma) : P(n) \to P(n)$ and $P(\tau) : P(X', x_0') \to P(X, x_0)$ respectively. We have so far in this chapter simply written $\sigma : P(n) \to P(n)$ and $\tau : P(X, x_0) \to P(X', x_0')$ to mean the same thing. Similarly, in this section we will require a bijection $\alpha : \underline{m}_* \to X$, and will denote the induced map $P(\alpha) : P(X, x_0) \to P(\underline{m}_*)$ simply by α.

2.4.1 Skeletal and non-skeletal composition operations

Before we describe the proof of the equivalence of definitions, let us highlight the key difference between them. The difference is the renumbering that is built into the skeletal partial composition operations. For $p \in P(m), q \in P(n)$, in order for $p \bullet_i q$ to lie in $P(m + n - 1)$, there has to be some renumbering of input labels. This is because $p \circ_i q$ has inputs with labels $\{1, 2, \dots, n\} \sqcup \{1, 2, \dots, m\} \setminus \{i\}$, so after composition there would be duplicate labels without such a map. Recall the renumbering map $\varphi : \Sigma_* \to \mathcal{B}ij_*$ given in Definition 2.1.1.1.

Explicitly, the skeletal composition operation is itself the composition $\bullet_i := \varphi_i \circ_i$ of $\circ_i : P(m) \otimes P(n) \to P(\underline{m}_* \sqcup_i \underline{n}_*)$ with the renumbering map $\varphi_i : \underline{m+n-1}_* \to \underline{m}_* \sqcup_i \underline{n}_*$. We view $\underline{m}_*$, $\underline{n}_*$ as objects of $\mathcal{B}ij_*$, with $i \in \underline{m}$. Then we have

$$\circ_i : P(\underline{m}_*) \otimes P(\underline{n}_*) \to P(\underline{m}_* \sqcup_i \underline{n}_*)$$

as structure of a non-skeletal operad. We obtain composition maps $\bullet_i$ via the diagram below.

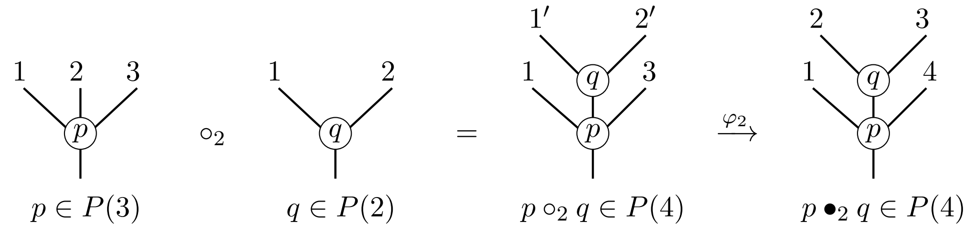
$$\begin{array}{ccc} E^{\#}P(m) \otimes P(n) & \xrightarrow{\bullet_i} & P(m+n-1) \\ \Big\| & & \uparrow{\scriptstyle \varphi_i} \\ P(\underline{m}_*) \otimes P(\underline{n}_*) & \xrightarrow[\circ_i]{} & P(\underline{m}_* \sqcup_i \underline{n}_*). \end{array} \tag{2.1}$$

The left equality is due to the inclusion $\Sigma_* \hookrightarrow \mathcal{B}ij_*$.

Example 2.4.1.1. Here is an example of the renumbering map with trees. Technically the labels after applying $\circ_2$ are the elements of the set given by

$$\{1, 2, 3\} \sqcup_2 \{1, 2\} = \{(1, 0), (3, 0), (1, 1), (2, 1)\},$$

but we simply write for example 1 for $(1, 0)$ and $1'$ for $(1, 1)$ to simplify notation.



2.4.2 Proof of equivalence

We will formulate a categorical proof of the equivalence of skeletal and non-skeletal operad definitions, similarly to that of Lukács in [Luk13].

First, we note that there is an equivalence $\mathcal{B}ij_* \simeq \Sigma_*$ as shown in Example 1.2.0.5 by the fact Σ_* is the skeleton of the category $\mathcal{B}ij_*$. We have the restriction and extension functions $R : \mathcal{B}ij_* \to \Sigma_*$ and $E : \Sigma_* \to \mathcal{B}ij_*$ respectively, with induced functors

$$E^* : Fun(\mathcal{B}ij_*^{op}, \mathcal{C}) \rightleftarrows Fun(\Sigma_*^{op}, \mathcal{C}) : R^*$$

on the functor categories of $\mathcal{C}$-collections. The induced functors R^* and E^* give an equivalence of the categories of collections, due to Proposition 1.2.0.8 and the fact that $\Sigma_*^{op} \simeq \mathcal{B}ij_*^{op}$ since the underlying categories are equivalent.

We will lift these to functors $R^{\#}$ and $E^{\#}$ on the operad categories $\mathcal{O}p_{\mathcal{B}ij_*}$ and $\mathcal{O}p_{\Sigma_*}$,

$$E^{\#} : \mathcal{O}p_{\mathcal{B}ij_*} \rightleftarrows \mathcal{O}p_{\Sigma_*} : R^{\#}.$$

Then we will show that the functor $E^{\#}$ is an equivalence of categories $\mathcal{O}p_{\mathcal{B}ij_*} \xrightarrow{\sim} \mathcal{O}p_{\Sigma_*}$.

Proposition 2.4.2.1. *The functors $R^{\#} : \mathcal{O}p_{\Sigma_*} \to \mathcal{O}p_{\mathcal{B}ij_*}$ and $E^{\#} : \mathcal{O}p_{\mathcal{B}ij_*} \to \mathcal{O}p_{\Sigma_*}$ on operads are well-defined.*

Proof. We have functors $R^{\#}$ and $E^{\#}$ on operads, agreeing with the functors R^*, E^* on the underlying collections. This means, we need to check compatibility with the composition operation, and that the equivariance, associativity and unit properties are preserved in each direction.

We will check that for both $R^{\#}$ and $E^{\#}$, applying the functor to an operad indeed gives us an operad in the target category. We will do these checks first on objects, and then on morphisms.

$E^{\#}$ is well-defined on objects

This is due to the inclusion $E : \Sigma_* \hookrightarrow \mathcal{B}ij_*$ and the fact that a Σ_*-operad is really a special case of a $\mathcal{B}ij_*$-operad. Let $P : \mathcal{B}ij_*^{op} \to \mathcal{C}$ be a $\mathcal{B}ij_*$-operad. Then

$$E^{\#}P(n) = E^{\#}(P(\underline{n}_*)) = P(\underline{n}_*).$$

Similarly, for $\sigma \in \Sigma_n$, we have $E^{\#}(P)(\sigma) = P(\sigma)$, since $\sigma \in Mor(\mathcal{B}ij_*)$. Define the partial composition operations

$$\bullet_i : E^{\#}P(m) \otimes E^{\#}P(n) \to E^{\#}P(m+n-1)$$

by composing composition $\circ_i$ that belongs to $\mathcal{O}p_{\mathcal{B}ij_*}$ with the renumbering map φ_i,

$$P(\underline{m}_*) \otimes P(\underline{n}_*) \xrightarrow{\circ_i} P(\underline{m}_* \sqcup_i \underline{n}_*) \xrightarrow{\varphi_i} E^{\#}P(m+n-1).$$

This gives us the complete operad structure. Finally, the equivariance, associativity and unit properties are inherited from the category $\mathcal{O}p_{\mathcal{B}ij_*}$.

$R^{\#}$ is well-defined on objects

Suppose that P is a Σ_*-operad. Then we wish to show that $R^{\#}P$ is a $\mathcal{B}ij_*$-operad.

We have the usual composition operations $\bullet_i$ in $\mathcal{O}p_{\Sigma_*}$, and also the composition operations $\circ_i = \varphi_i^{-1}\bullet_i$ in $\mathcal{O}p_{\mathcal{B}ij_*}$ as discussed earlier. That is,

$$\circ_i : P(m) \otimes P(n) \to P(\underline{m}_* \sqcup_i \underline{n}_*).$$

We then have the following commutative diagram as a result of equivariance in $\mathcal{O}p_{\Sigma_*}$ and properties of the renumbering map. In particular, the outer square is an equivariance condition for the composition operation for skeletal operads without the renumbering map, which gives an operad in $\mathcal{O}p_{\mathcal{B}ij_*}$ rather than $\mathcal{O}p_{\Sigma_*}$. For $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$, we have

$$\begin{array}{ccccc} & & \xrightarrow{\qquad\qquad\circ_{\sigma(i)}\qquad\qquad} & & \\ P(m) \otimes P(n) & \xrightarrow{\bullet_{\sigma(i)}} & P(m+n-1) & \xrightarrow{\varphi^{-1}_{\sigma(i)}} & R^{\#}P(\underline{m}_* \sqcup_{\sigma(i)} \underline{n}_*) \\ \Big\downarrow{\scriptstyle \sigma\otimes\tau} & & \Big\downarrow{\scriptstyle \sigma\bullet_i\tau} & & \Big\downarrow{\scriptstyle \sigma\circ_i\tau} \\ P(m) \otimes P(n) & \xrightarrow[\bullet_i]{} & P(m+n-1) & \xrightarrow[\varphi_i^{-1}]{} & R^{\#}P(\underline{m}_* \sqcup_i \underline{n}_*) \\ & & \xrightarrow[\qquad\qquad\circ_i\qquad\qquad]{} & & \end{array} \tag{2.2}$$

We wish to construct a similar diagram which patches together the diagram above with a non-skeletal analogue.

Let X, Y be sets in $\mathcal{B}ij_*$, $x \in X \setminus \{x_0\}$ and $1 \leq i \leq m$. Choose maps $\alpha : \underline{m}_* \to X$, $\beta : \underline{n}_* \to Y \in \mathcal{B}ij_*$, such that $\alpha(i) = x$. Then define the non-skeletal composition $\circ_x$ by commutativity of the diagram

$$\begin{array}{ccc} R^{\#}P(X, x_0) \otimes R^{\#}P(Y, y_0) & \xrightarrow{\circ_x} & R^{\#}P(X \sqcup_x Y, x_0) \\ \Big\downarrow{\scriptstyle \alpha\otimes\beta} & & \Big\downarrow{\scriptstyle \alpha\circ_i\beta} \\ P(m) \otimes P(n) & \xrightarrow[\circ_i]{} & R^{\#}P(\underline{m}_* \sqcup_i \underline{n}_*). \end{array} \tag{2.3}$$

Lukács shows that this is well-defined and independent of choice of α, β.

Then it remains to check that the $\mathcal{B}ij_*$-operad $R^{\#}P$ satisfies the equivariance, associativity and unit axioms.

In each case, the check is similar. It involves patching together commutative diagrams and checking commutativity of the relevant square for the condition we are interested in. Here we will check the equivariance case, and omit the checks for associativity and unit.

(Equivariance): Let us patch some diagrams together to get the following large diagram. Then the inner square commuting is the equivariance condition we need.

$$
\begin{array}{ccccc}
 & P(m) \otimes P(n) & \xrightarrow{\quad \circ_j \quad} & R^{\#}P(\underline{m}_* \sqcup_j \underline{n}_*) & \\
 & \uparrow{\scriptstyle \alpha' \otimes \beta'} & & \uparrow{\scriptstyle \alpha' \circ_j \beta'} & \\
 & R^{\#}P(X', x_0') \otimes P(Y', y_0') & \xrightarrow{\circ_{\rho(x)}} & R^{\#}P(X' \sqcup_{\rho(x)} Y', x_0') & \\
{\scriptstyle (\alpha')^{-1}\rho\alpha \otimes (\beta')^{-1}\pi\beta} & \downarrow{\scriptstyle \rho \otimes \pi} & & \downarrow{\scriptstyle \rho \circ_x \pi} & {\scriptstyle (\alpha')^{-1}\rho\alpha \circ_j (\beta')^{-1}\pi\beta} \\
 & R^{\#}P(X, x_0) \otimes P(Y, y_0) & \xrightarrow{\quad \circ_x \quad} & R^{\#}P(X \sqcup_x Y, x_0) & \\
 & \downarrow{\scriptstyle \alpha \otimes \beta} & & \downarrow{\scriptstyle \alpha \circ_i \beta} & \\
 & P(m) \otimes P(n) & \xrightarrow[\quad \circ_i \quad]{} & R^{\#}P(\underline{m}_* \sqcup_i \underline{n}_*). &
\end{array}
$$

We have the maps $\alpha : \underline{m}_* \to X, \beta : \underline{n}_* \to Y$ with $\alpha(i) = x$. Similarly, we have maps $\alpha' : \underline{m}_* \to X', \beta' : \underline{n}_* \to Y'$. These are the chosen maps $\Sigma_* \to \mathcal{B}ij_*$ as in Example 1.2.0.5. Let $\pi : X \to X'$, and $\tau : Y \to Y'$ be bijections. Finally, we define $j := (\alpha')^{-1}(\rho(x))$, so that $\alpha'(j) = \rho(x)$.

All the vertical arrows in the diagram are bijections and hence invertible maps.

Both the top and bottom squares in this diagram commute as they are instances of diagram (2.3), that is, the definition of the $\circ_x$ composition. The left and right vertical maps are by definition such that the left and right squares commute. Then the outer square is an example of the equivariance diagram (2.2) for the skeletal operad P without the renumbering map, and thus commutes as well.

Therefore, the inner square commutes as required.

Operad maps

Now, we need to show that $E^{\#}$ and $R^{\#}$ are well-defined on morphisms, that is, operad maps.

In particular, given an operad map $\theta : P \to Q$ of $\mathcal{B}ij_*$-operads, we may choose a bijection $\alpha : \underline{m}_* \to (X, x_0)$ as in 1.2.0.5 such that $E^{\#}\theta$ is defined by the following diagram.

$$
\begin{array}{ccc}
P(X, x_0) & \xrightarrow{\theta_X} & Q(X, x_0) \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\
E^{\#}P(m) & \xrightarrow[E^{\#}\theta_m]{} & E^{\#}Q(m).
\end{array}
\tag{2.4}
$$

Similarly, given a map $\Phi : P \to Q$ of Σ_*-operads, the map $R^{\#}\Phi$ is defined by the diagram below.

$$\begin{array}{ccc} R^{\#}P(X,x_0) & \xrightarrow{R^{\#}\Phi_X} & R^{\#}Q(X,x_0) \\ \alpha\big\downarrow & & \big\downarrow\alpha \\ P(m) & \xrightarrow[\Phi_m]{} & Q(m). \end{array} \tag{2.5}$$

The operad maps θ and Φ satisfy the conditions shown by the diagrams in 2.3.0.2 and 2.2.0.5 respectively. That is, compatibility with the group action, composition operations, and unit.

$E^{\#}$ is well-defined on morphisms

If θ is a map of $\mathcal{B}ij_*$-operads, then $E^{\#}(\theta)$ can also be viewed as a map of Σ_* operads via the inclusion $\Sigma_* \hookrightarrow \mathcal{B}ij_*$. Using the fact that $E^{\#}(\theta)_n = \theta_{\underline{n}_*}$, the commutativity of the diagrams follow from the commutativity of the non-skeletal analogues.

$R^{\#}$ is well-defined on morphisms

If conversely Φ is a map of Σ_* operads, we assume the diagrams for compatibility with permutations in Σ_*, composition $\bullet_i$ and unit. Then it remains to show that the equivalent diagrams for $R^{\#}\Phi$ commute. That is, that we have compatibility with bijections in $\mathcal{B}ij_*$, the composition operations $\circ_x$, and unit.

- (Compatibility with bijections) The following diagram commutes due to equivariance of Φ on skeletal operads.

  $$\begin{array}{ccc} P(m) & \xrightarrow{\Phi_m} & Q(m) \\ \sigma\big\downarrow & & \big\downarrow\sigma \\ P(m) & \xrightarrow[\Phi_m]{} & Q(m) \end{array}$$

  Then the relevant diagram for $R^{\#}\Phi$ that we want to commute, for $\rho : X \to X'$, is

  $$\begin{array}{ccc} R^{\#}P(X',x_0') & \xrightarrow{R^{\#}\Phi_{X'}} & R^{\#}Q(X',x_0') \\ \rho\big\downarrow & & \big\downarrow\rho \\ R^{\#}P(X,x_0) & \xrightarrow[R^{\#}\Phi_X]{} & R^{\#}Q(X,x_0). \end{array}$$

  Therefore, we require that the middle square of the larger diagram below commutes.

$$\begin{array}{ccccc}
 & P(m) & \xrightarrow{\Phi_m} & Q(m) & \\
 & \big\uparrow{\scriptstyle \alpha'} & & \big\uparrow{\scriptstyle \alpha'} & \\
 & R^{\#}P(X', x_0') & \xrightarrow{R^{\#}\Phi_{X'}} & R^{\#}Q(X', x_0') & \\
\alpha\rho(\alpha')^{-1} & \big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \rho} & \alpha\rho(\alpha')^{-1} \\
 & R^{\#}P(X, x_0) & \xrightarrow[R^{\#}\Phi_X]{} & R^{\#}Q(X, x_0) & \\
 & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha} & \\
 & P(m) & \xrightarrow[\Phi_m]{} & Q(m). &
\end{array}$$

The top and bottom squares commute by definition of the map $R^{\#}$, given by diagram (2.5). The outer square commutes due to compatibility of Φ with permutations in Σ_*, since Φ is a morphism in $\mathcal{O}p_{\Sigma_*}$. The left and right squares commute by construction. Therefore, the middle square commutes as required.

- (Compatibility with composition) We assume commutativity of the diagram below for Φ in the skeletal case.

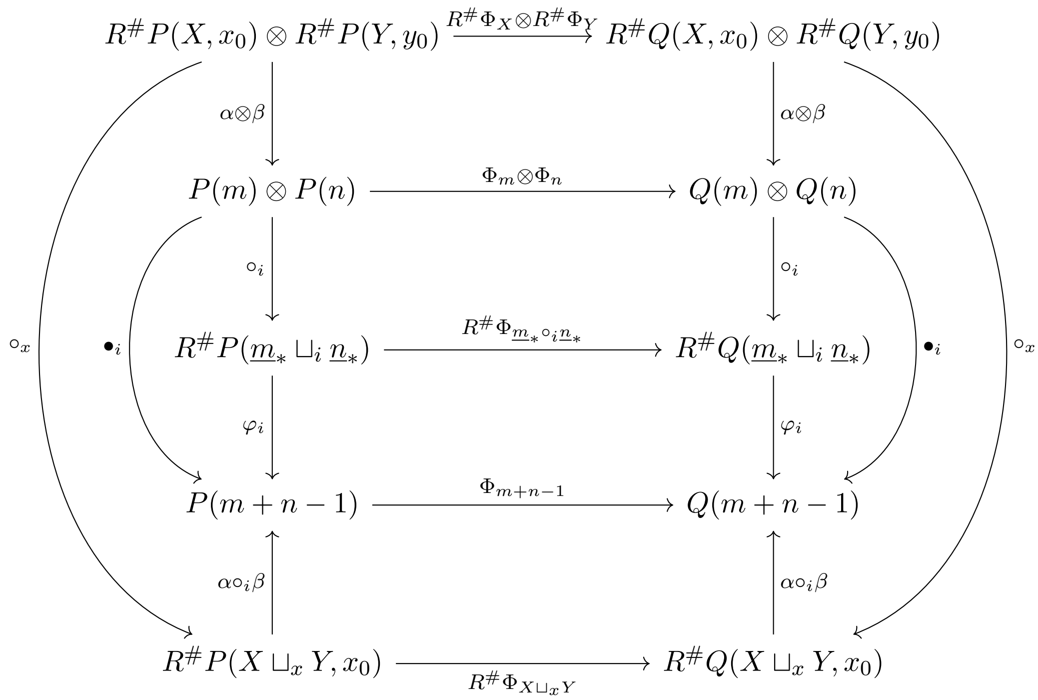
$$\begin{array}{ccccc}
 & P(m) \otimes P(n) & \xrightarrow{\Phi_m \otimes \Phi_n} & Q(m) \otimes Q(n) & \\
 & \big\downarrow{\scriptstyle \circ_i} & & \big\downarrow{\scriptstyle \circ_i} & \\
\bullet_i & R^{\#}P(\underline{m}_* \sqcup_i \underline{n}_*) & \xrightarrow{R^{\#}\Phi_{\underline{m}_* \sqcup_i \underline{n}_*}} & R^{\#}Q(\underline{m}_* \sqcup_i \underline{n}_*) & \bullet_i \\
 & \big\downarrow{\scriptstyle \varphi_i} & & \big\downarrow{\scriptstyle \varphi_i} & \\
 & P(m+n-1) & \xrightarrow[\Phi_{m+n-1}]{} & Q(m+n-1) &
\end{array} \tag{2.6}$$

These two squares correspond to the square in Definition 2.2.0.5, with the $\bullet_i$ operation separated into $\circ_i$ and the renumbering map φ_i.

The relevant diagram required to commute in order for $R^{\#}\Phi$ to be a map of non-skeletal operads is

$$\begin{array}{ccc}
R^{\#}P(X, x_0) \otimes R^{\#}P(Y, y_0) & \xrightarrow{R^{\#}\Phi_X \otimes R^{\#}\Phi_Y} & R^{\#}Q(X, x_0) \otimes R^{\#}Q(Y, y_0) \\
\big\downarrow{\scriptstyle \circ_x} & & \big\downarrow{\scriptstyle \circ_x} \\
R^{\#}P(X \sqcup_x Y, x_0) & \xrightarrow[R^{\#}\Phi_{X \sqcup_x Y}]{} & R^{\#}Q(X \sqcup_x Y, x_0)
\end{array}$$

We can patch together these diagrams using maps $\alpha : \underline{m}_* \to (X, x_0), \beta : \underline{n}_* \to (Y, y_0)$, then we get the following. The above diagram, which we require to commute in order to have compatibility with composition in the non-skeletal case, is the outer square in the following diagram.



We assume commutativity of the middle two squares, which are simply instances of diagram (2.6). Then the top and bottom square commute by definition of the map $R^{\#}\Phi$ (see diagram (2.5)). The left and right arrows are such that the left and right squares commute. Therefore, the outer square commutes as required.

The unit check involves a similar process to those above, so we omit that here.

Therefore, we have that $R^{\#}\Phi$ is indeed a map of $\mathcal{B}ij_*$-operads.

□

Lemma 2.4.2.2. *The functor $E^{\#}$ is essentially surjective.*

Proof. This is by the inclusion $E:\Sigma_*\hookrightarrow \mathcal{B}ij_*$ and the discussion in Proposition 2.4.2.1. For a skeletal operad $P\in\mathcal{O}p_{\Sigma_*}$, the operad $R^{\#}P\in\mathcal{O}p_{\mathcal{B}ij_*}$ is such that $E^{\#}R^{\#}P=P$. □

Lemma 2.4.2.3. *The functor $E^{\#}$ is fully faithful.*

Proof. We show that the functor $E^*: Fun(\mathcal{B}ij^{op},\mathcal{C})\to Fun(\Sigma^{op},\mathcal{C})$ is fully faithful. We have that for a morphism Φ of Σ_*-operads, $E^{\#}R^{\#}\Phi=\Phi$, meaning $E^{\#}$ is full.

For operad maps $\theta_1,\theta_2 : P\to Q$ with P and Q objects of $\mathcal{O}p_{\mathcal{B}ij_*}$, we have that $E^{\#}(\theta_1)_n=(\theta_1)_{\underline{n}_*}$ and $E^{\#}(\theta_2)_n=(\theta_2)_{\underline{n}_*}$. Therefore, if $E^{\#}\theta_1=E^{\#}\theta_2$ then we have $\theta_1=\theta_2$.

□

Theorem 2.4.2.4 ([Luk13, Theorem 4.4]). *There is an equivalence of categories* $E^{\#} : \mathcal{O}p_{\mathcal{B}ij_*} \to \mathcal{O}p_{\Sigma_*}$.

Proof. By Lemmas 2.4.2.2 and 2.4.2.3, and the well known result [Lei14, Proposition 1.3.18], $E^{\#}$ gives an equivalence of categories $E^{\#} : \mathcal{O}p_{\mathcal{B}ij_*} \xrightarrow{\sim} \mathcal{O}p_{\Sigma_*}$. □

We therefore have the following commutative diagram showing the equivalence of operad categories, with forgetful functors U to the associated functor categories.

$$\begin{array}{ccc} \mathcal{O}p_{\mathcal{B}ij_*} & \underset{R^{\#}}{\overset{E^{\#}}{\rightleftarrows}} & \mathcal{O}p_{\Sigma_*} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ Fun(\mathcal{B}ij_*^{op}, \mathcal{C}) & \underset{R^{*}}{\overset{E^{*}}{\rightleftarrows}} & Fun(\Sigma_*^{op}, \mathcal{C}) \end{array}$$

Remark 2.4.2.5. One could also obtain the result of Theorem 2.4.2.4 by showing that the forgetful functor U is faithful, and then the commutativity of the diagram above gives the necessary properties for $E^{\#}$.

2.5 Non-symmetric operads

So far in this chapter and throughout the thesis, we mainly consider symmetric operads. That is, those with an action of the symmetric group Σ_n on the n^{th} arity $P(n)$ of the operad, and the composition operation satisfying the equivariance axiom.

We may define a non-symmetric operad by removing the equivariance axiom. This means that any symmetric operad has an underlying non-symmetric operad by forgetting the symmetric group action, and conversely a non-symmetric operad gives rise to a symmetric operad by taking for each $n \geq 1$, the product $P(n) := P(n) \times \Sigma_n$, together with induced composition operations.

Definition 2.5.0.1 (Non-symmetric operad [MSS02]). A non-symmetric operad in $\mathcal{C}$ is a sequence $\{P(n)\}_{n>0}$ together with a unit $1 \in P(1)$ and a partial composition operation

$$\bullet_i : P(m) \times P(n) \to P(m+n-1)$$

for $m, n > 0$ and $1 \leq i \leq m$, satisfying associativity and unit axioms.

Where rooted trees were used to illustrate the symmetric operad setting, one can similarly use planar rooted trees to illustrate non-symmetric operads. In fact, the operad of planar rooted trees gives the free non-symmetric operad, as discussed in [LV12, p. 5.8.6].

Remark 2.5.0.2. We had that a symmetric operad can be viewed as a monoid in the category of symmetric sequences. Similarly, a non-symmetric operad can be viewed as a monoid in the category of sequences (that are not required to be symmetric) with composition operations.

2.6 Examples

There are a few fundamental operads which we will mention here as examples. Many other useful examples are given in [MSS02], [Cha08], and [LV12].

Definition 2.6.0.1 (Topological operad)**.** Topological operads are operads in the category Top of topological spaces and continuous maps, or the category Top_* of based topological spaces and basepoint preserving continuous maps.

Example 2.6.0.2 (The endomorphism operad [LV12, p. 5.2.12])**.** The endomorphism operad is one of the most fundamental operads, in particular it is often used to describe algebras over operads, as mentioned in Remark 2.7.0.2.

For a vector space V over a field k, the endomorphism operad End_V over V is given by

$$End_V(n) := Hom(V^{\otimes n}, V).$$

The symmetric group Σ_n acts on $V^{\otimes n}$ by permutation, and this induces the right Σ_n action on $End_V(n)$. Composition is given by composition of endomorphisms.

Example 2.6.0.3 (The free operad on multiplication)**.** The free operad $P(\mu)$ is the operad with one operation on each level $P(\mu)(n)$, that is multiplication μ. This is the simplest operad which has the least structure.

Example 2.6.0.4 (The commutative operad)**.** The commutative operad is the operad of k-vector spaces in which the operations are commutative multiplication in k. We have

$$\mathcal{C}omm(n) := k \quad \text{for all } n \geq 0.$$

The symmetric group Σ_n acts trivially on $\mathcal{C}omm(n)$.

The following definition is a property that a number of operads that we consider have.

Example 2.6.0.5 (The symmetric associative operad, $\mathcal{A}ss_\Sigma$ [MSS02])**.** The associative operad is another fundamental and important operad, where the operations are simply associative multiplication. This operad is used to study associative algebras. We will return to this example in later chapters to further explore its structure.

The symmetric operad $\mathcal{A}ss_\Sigma$ in Mod_k is defined by

$$\mathcal{A}ss_\Sigma(n) \cong k[\Sigma_n] = k\{\mu_n\pi | \pi \in \Sigma_n\},$$

with the associative multiplication operation

$$\mu_n = \mu(\mu \otimes 1)...(\mu \otimes 1 \otimes ... \otimes 1),$$

and action of Σ_n on $\mathcal{A}ss_\Sigma(n)$ by permuting the positions in multiplication as follows,

$$\mu_n(a_1, a_2, \dots, a_n)\pi = \mu_n(a_{\pi^{-1}(1)}, a_{\pi^{-1}(2)}, \dots, a_{\pi^{-1}(n)}).$$

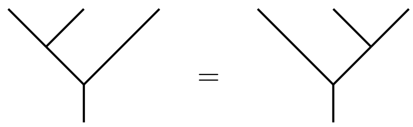
We have the associativity condition

$$\mu(\mu \otimes id) := \mu(id \otimes \mu),$$

which can equivalently be formulated in terms of the partial composition operations $\bullet_i$ as

$$\mu \bullet_1 \mu := \mu \bullet_2 \mu.$$

This can be depicted using tree diagrams.



These are operations in $\mathcal{A}ss_\Sigma(3)$ as the composition of operations in $\mathcal{A}ss_\Sigma(2)$. We may define the associativity condition for elements of $\mathcal{A}ss_\Sigma(2)$ because this determines associativity of operations in $\mathcal{A}ss_\Sigma(n)$ for larger n via the operad composition maps.

The action of the symmetric group Σ_n on $\mathcal{A}ss_\Sigma(n)$ is that associated to the regular representation, given by permutation of inputs.

Example 2.6.0.6 (The non-symmetric associative operad)**.** This is the operad for associative multiplication, that has no Σ_n action on $\mathcal{A}ss(n)$.

The non-symmetric associative operad is defined by

$$\mathcal{A}ss(n) := k, \qquad \text{for all } n \geq 1.$$

By taking the product with the symmetric group, we obtain the symmetric associative operad. That is, we have

$$\mathcal{A}ss_\Sigma(n) = \mathcal{A}ss(n) \times \Sigma_n.$$

Example 2.6.0.7 (The Lie operad)**.** At the n^{th} arity, $\mathcal{L}ie(n)$ is given by the k-linear combinations of all complete bracketings of symbols $x_1, x_2, \ldots, x_n$, with no repetitions. For example,

$$[\ldots[[x_n, x_{n-1}]x_{n-2}]\ldots x_1] \in \mathcal{L}ie(n).$$

One can also generate $\mathcal{L}ie(n)$ from the skew symmetric operation in $\mathcal{L}ie(2)$:

$$\mathcal{L}ie(2) := [-,-]$$

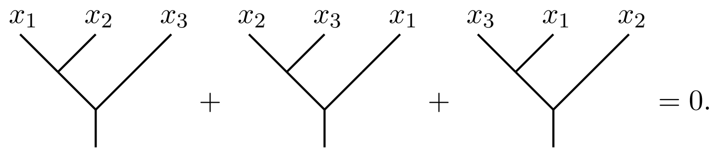
where $[-,-]$ is the Lie bracket. Higher operations are generated by composition, where composition is given by nesting Lie brackets. For example, for $p = [x_1, x_2] \in \mathcal{L}ie(2)$, $q = [x_1, x_2] \in \mathcal{L}ie(2)$, we have

$$p \bullet_1 q := [[x_1, x_2], x_3],$$

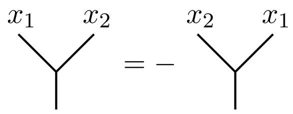
subject to the Jacobi relation

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$

As with the associative operad, we may express the equivalence relations using tree diagrams. The Jacobi identity is shown diagrammatically below.



Skew symmetry can be depicted similarly:



The Lie operad is symmetric, and is a suboperad of the associative operad. Analogously to the associative operad, there is a non-symmetric Lie operad, that is a suboperad of the non-symmetric associative operad $\mathcal{A}ss$.

The Lie operad is also the Koszul dual of the commutative operad, as explained in [GK94].

2.7 Operad algebras

One of the main purposes of operads is to study the algebras over them. For example, we obtain associative, commutative and Lie algebras from their associated operads. Below is the definition given by Loday and Vallette.

Definition 2.7.0.1 (Operad algebra [LV12, Section 5.2.3])**.** An operad algebra or an algebra over an operad P is given by a vector space A, and a map $\gamma_A : P(A) \to A$ such that the two diagrams below commute.

$$\begin{array}{ccccc} (P \circ P)(A) & = & P(P(A)) & \xrightarrow{P(\gamma_A)} & P(A) \\ \downarrow{\scriptstyle \gamma(A)} & & & & \downarrow{\scriptstyle \gamma_A} \\ P(A) & & \xrightarrow[\gamma_A]{} & & A \end{array} \qquad \begin{array}{ccc} I(A) & \xrightarrow{\eta(A)} & P(A) \\ & \searrow^{\simeq} & \downarrow{\scriptstyle \gamma_A} \\ & & A \end{array}$$

The collection of algebras over an operad P forms a category, with morphisms $f : A \to A'$ of algebras A and A' linear maps such that the following diagram commutes.

$$\begin{array}{ccc} P(A) & \xrightarrow{\eta_A} & A \\ \downarrow{\scriptstyle P(f)} & & \downarrow{\scriptstyle f} \\ P(A') & \xrightarrow[\eta_{A'}]{} & A' \end{array}$$

Remark 2.7.0.2. An algebra over an operad P on a vector space V can also be described as a morphism $P \to End_V$.

Remark 2.7.0.3. One can also define a coalgebra over an operad.

2.8 Operad (co)homology

Recall from Definition 2.6.0.1 that a topological operad is an operad in the category Top_* or Top.

Definition 2.8.0.1 (Homology of a topological operad, [Chi05, Definition 9.3])**.** For an operad $P \in Top$ or $P \in Top_*$, the homology H_*P is given by

$$(H_*P)(n) := H_*(P(n)).$$

Then H_*P is an operad in Mod_k, as shown by Ching in [Chi05, Lemma 9.4].

Similarly, one can take the cohomology H^*P of an operad, resulting in a cooperad in Mod_k. This is subject to some flatness and finiteness conditions, as explained by Ching in [Chi05, Remark 9.5].

2.9 Operadic suspension

Finally, we will give one more property that will be of use in Chapter 6. Operadic suspension is a type of suspension specific to operads.

Definition 2.9.0.1 (Operadic (de)suspension, [GK95, Section 2.10])**.** The suspension of a dg-operad P is the operad $\mathfrak{s}P$ defined by

$$\mathfrak{s}P(n) := \Sigma^{n-1} sgn \otimes P(n),$$

where sgn is the sign representation of Σ_n. Then the desuspension of an operad P is inverse to the suspension. That is,

$$\mathfrak{s}^{-1}P(n) := \Sigma^{1-n} sgn \otimes P(n).$$

As we would hope, for an algebra A over P, it is true that ΣA is an algebra over $\mathfrak{s}P$.

Note that the operadic suspension and desuspension are sometimes defined the other way around in the literature.

Chapter 3

Cyclic operads

A cyclic operad is an operad that has some extra structure. That extra structure is in the form of a Σ_{n+1} action on the n^{th} arity $P(n)$ of the operad. What this looks like is being able to freely permute the output of an operation with the inputs. This may seem unintuitive, but the concept was introduced by Getzler and Kapranov as a way to generalise the use of cyclic homology to other algebra structures. Cyclic homology was itself developed for the study of non-commutative, associative algebras. Cyclic operads also allow one to study the invariant bilinear forms on algebras over operads.

As with general operads, there are a number of definitions in the literature of a cyclic operad. One such definition is to view a cyclic operad as an ordinary operad with extra structure. This is done by extending the action of Σ_n on $P(n)$ to an action of Σ_{n+1} by giving the explicit action of a suitable generating permutation.

We will prove the analogous equivalence of skeletal and non-skeletal definitions in this setting, following the same structure of proof as for operads in Chapter 2. Later in the chapter, we provide some key examples and non-examples of cyclic operads, as well as discuss briefly some properties of cyclic operads.

3.1 Background

Throughout this chapter we will work with the category Σ of based sets $\underline{n}_*$ and unbased permutations in Σ_{n+1}, and the category $\mathcal{B}ij$ of based finite sets and unbased bijections. See example 1.2.0.6 for the relationship between these categories. We call the root label 0 following the same convention as Getzler and Kapranov [GK95] and using the bijection $\Sigma_{n+1} \to \Sigma_{\underline{n}_*}$ where $\underline{n}_* := \{0, 1, \ldots, n\}$.

We will also use the same deleted disjoint union given by Definition 2.1.0.1. We still work with based sets here because we require a distinguished root label, since we view cyclic operads as operads with extra structure.

3.1.1 Renumbering map

We can view the renumbering map $\varphi_i : \mathcal{B}ij_* \to \Sigma_*$ given in Definition 2.1.1.1 as a map $\varphi_i : \mathcal{B}ij \to \Sigma$, and it behaves exactly as in the case of non-cyclic operads. The extra structure of a cyclic operad involves only the interaction of the composition operation with the extra action of the symmetric group.

3.1.2 Extending the symmetric group action

In this chapter, we will work with a generating permutation that extends the action of Σ_n to Σ_{n+1}. Indeed, any permutation in Σ_{n+1} that permutes the extra element 0, will work. Getzler and Kapranov use the $(n+1)$-cycle $(0, 1, \dots, n)$ in [GK95, Theorem 2.2], however we will show the definition of a cyclic operad can equivalently be formulated using the transposition $(0, 1)$ as suggested by Obradović [Obr17]. See section 1.1 for the details.

When permuting the root and composing operations, we sometimes change the order of composition. Recall from (1.1) that for P an operad in $\mathcal{C}$, we have the map t

$$t : P(m) \otimes P(n) \to P(n) \otimes P(m)$$
$$p \otimes q \mapsto q \otimes p,$$

that swaps the order of operations, using the symmetric monoidal structure of $\mathcal{C}$.

3.2 Skeletal cyclic operad definitions

The following definition using $(n+1)$-cycles is the standard definition of a skeletal cyclic operad as introduced by Getzler and Kapranov. Recall from Definition 2.2.0.1 that a skeletal operad P is a collection $\Sigma_*^{op} \to \mathcal{C}$ with composition operations. Since this is a functor from the opposite category, the product $\sigma\tau$ of permutations means that σ acts first and then τ.

Definition 3.2.0.1 (Cyclic skeletal operad [MSS02, Definition 5.2])**.** A skeletal operad P together with an extension of the symmetric group action on $P(n)$ from Σ_n to Σ_{n+1}, subject to the below conditions, is a cyclic skeletal operad. For $1 \in P(1)$ the unit of the operad P, $p \in P(m)$, $q \in P(n)$, $2 \leq i \leq m$, and τ_k the cycle $(0, 1, ..., k) \in \Sigma_{k+1}$,

1. $(1)\tau_1 = 1$,
2. $(p \bullet_1 q)\tau_{m+n-1} = (q\tau_n) \bullet_n (p\tau_m)$,
3. $(p \bullet_i q)\tau_{m+n-1} = (p\tau_m) \bullet_{i-1} q$.

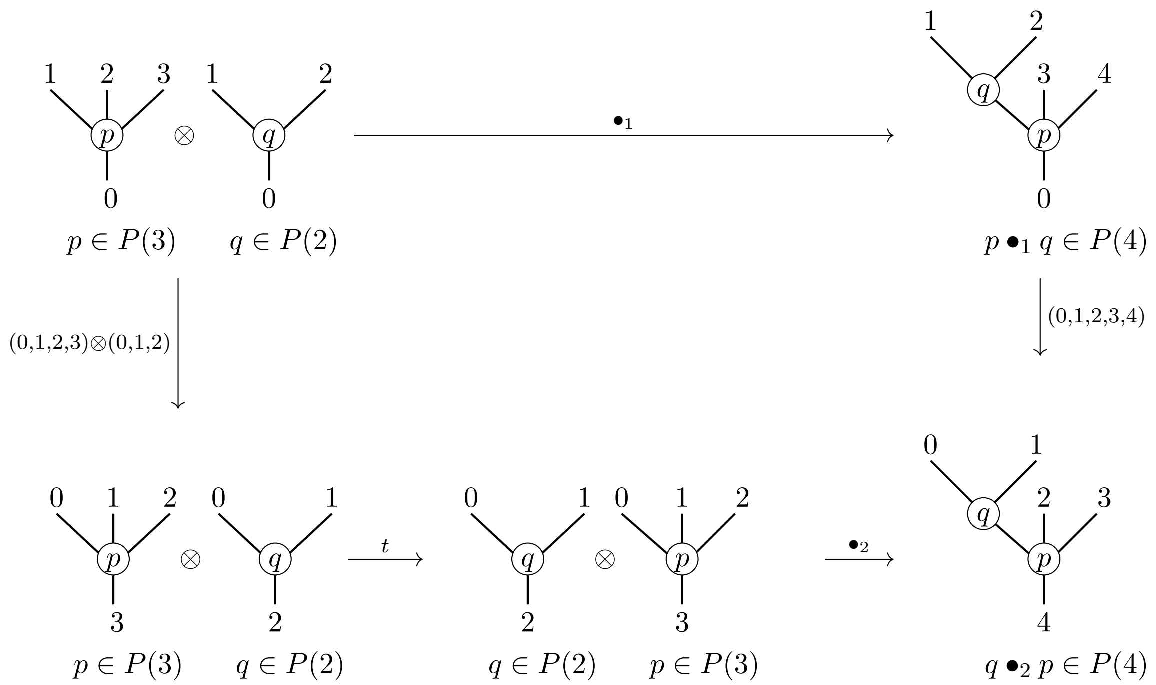
Remark 3.2.0.2. Note that definitions by Markl, Schnider and Stasheff such as in [MSS02], [Mar99] and [Mar08] have one more condition than the definition in [GK95]. This extra condition (condition 3 in Definition 3.2.0.1) is the one that specifies what happens when the root label is permuted but does not swap from p to q, and therefore does not change the order of composition.

It has been commented that this condition may follow automatically in some cases, as suggested by van der Laan in [Laa04], but this has not been shown to be true in general, so we include it here.

The second and third conditions give equivariance for the extra generating cycle, which can also be specified with commutative diagrams.

$$\begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_1} & P(m+n-1) \\ \downarrow{\scriptstyle \tau_m\otimes\tau_n} & & \\ P(m)\otimes P(n) & & \downarrow{\scriptstyle \tau_{m+n-1}} \\ \downarrow{\scriptstyle t} & & \\ P(n)\otimes P(m) & \xrightarrow[\bullet_n]{} & P(m+n-1) \end{array}
\qquad
\begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_i} & P(m+n-1) \\ \downarrow{\scriptstyle \tau_m\otimes id} & & \downarrow{\scriptstyle \tau_{m+n-1}} \\ P(m)\otimes P(n) & \xrightarrow[\bullet_{i-1}]{} & P(m+n-1) \end{array}$$

The diagram below is an example of the first diagram above, that is the second cyclic condition, depicted with trees. The commutativity of the diagram is simply equivariance of the composition operation in this case.



The associativity and unit axioms are the same as in the non-cyclic case, as they do not involve action of the symmetric group.

It will be useful to consider an alternative definition using the transposition $(0, 1)$ instead, as proposed by Obradović [Obr17]. This is because when it comes to comparing with a non-skeletal definition, the $(n + 1)$-cycle does not make sense in that setting where we do not have a canonical ordering of set elements.

Proposition 3.2.0.3. *The following definition of a cyclic operad using transpositions instead of $(n+1)$-cycles is equivalent to Definition 3.2.0.1.*

A cyclic operad P is an operad P with an extension of the Σ_n-action on $P(n)$ to an action of Σ_{n+1} such that for $1 \in P(1)$ the unit of P, $p \in P(m)$, $q \in P(n)$,

1. $(1)(0,1) = 1$,

2. $(p \bullet_1 q)(0,1)\phi = q(0,1) \bullet_1 p(0,1)$,

3. $(p \bullet_i q)(0,1) = p(0,1) \bullet_i q$.

In the above, $\phi \in \Sigma_{m+n-1}$ is the permutation given by

$$\begin{pmatrix} 1 & 2 & 3 & \dots & m & m+1 & \dots & m+n-1 \\ 1 & n+1 & n+2 & \dots & m+n-1 & 2 & \dots & n \end{pmatrix}.$$

Proof. Note that the unit conditions are the same in the two formulations.

Then for the other two conditions, we will use the fact that

$$\tau_n := (0,1,2,\dots,n) = (0,1)(1,2,\dots,n),$$

and we therefore have

$$(p \bullet_i q)(0,1) = (p \bullet_i q)\tau_{m+n-1}(1,2,\dots,m+n-1)^{-1},$$
$$(p \bullet_i q)\tau_{m+n-1} = (p \bullet_i q)(0,1)(1,2,\dots,m+n-1).$$

Cyclic condition 2

First let us assume P satisfies condition 2 in Definition 3.2.0.1. Let $p \in P(m)$, $q \in P(n)$. Then we have

$$\begin{aligned} q(0,1) \bullet_1 p(0,1) &= q\tau_n(1,2,\dots,n)^{-1} \bullet_1 p\tau_m(1,2,\dots,m)^{-1} \\ &= (q\tau_n \bullet_n p\tau_m)((1,2,\dots,n)^{-1} \bullet_1 (1,2,\dots,m)^{-1}) \\ &= (p \bullet_1 q)\tau_{m+n-1}((1,2,\dots,n)^{-1} \bullet_1 (1,2,\dots,m)^{-1}) \\ &= (p \bullet_1 q)(0,1)\phi. \end{aligned}$$

Here we used the equivariance property of an operad, that is that

$$(\sigma \circ_i \tau)(\bullet_i) = (\bullet_{\sigma(i)})(\sigma \otimes \tau)$$

for permutations of inputs σ and τ.

The last equality can be checked, using the fact that

$$\sigma \bullet_i \tau = (\varphi_{\sigma(i)})^{-1}(\sigma \circ_i \tau)(\varphi_i),$$

where φ_i, denotes the renumbering map.

Conversely, let us assume that P satisfies condition 2 in the statement of this proposition. Let $p \in P(m), q \in P(n)$. Then we have that

$$\begin{aligned} p\tau_m \bullet_n q\tau_n &= p(0,1)(1,2,\dots,m) \bullet_n q(0,1)(1,2,\dots,n) \\ &= (p(0,1) \bullet_1 q(0,1))((1,2,\dots,m) \bullet_m (1,2,\dots,n)) \\ &= (q \bullet_1 p)(0,1)\phi((1,2,\dots,n) \bullet_n (1,2,\dots,m)) \\ &= (p \bullet_1 p)\tau_{m+n-1}. \end{aligned}$$

We have again used equivariance of the underlying operad, as well as the fact that

$$\tau_{m+n-1} = (0,1)\phi((1,2,\dots,n) \bullet_n (1,2,\dots,m)).$$

Therefore, the second conditions agree.

Cyclic condition 3

First assume condition 3 in the statement of this proposition, and $i \neq 1$. We have

$$\begin{aligned} (p \bullet_i q)(0,1) &= (p \bullet_i q)\tau_{m+n-1}(1,2,\dots,m+n-1)^{-1} \\ &= ((p\tau_m) \bullet_{i-1} q)(1,2,\dots,m+n-1)^{-1}. \end{aligned}$$

Let $\sigma = (1,2,\dots,m)^{-1}$ and $\tau = id$. Then we have

$$\begin{aligned} ((p\tau_m) \bullet_{i-1} q)(1,2,\dots,m+n-1)^{-1} &= (p\tau_m(1,2,\dots,m)^{-1}) \bullet_i q \\ &= p(0,1) \bullet_i q. \end{aligned}$$

Conversely, assuming condition 3 in Proposition Skeletal cycle, we have

$$\begin{aligned} (p \bullet_i q)\tau_{m+n-1} &= (p \bullet_i q)(0,1)(1,2,\dots,m+n-1) \\ &= (p(0,1) \bullet_i q)(1,2,\dots,m+n-1). \end{aligned}$$

Let $\sigma = (1,2,\dots,m)$ and $\tau = id$. Then

$$\begin{aligned} (p(0,1) \bullet_i q)(1,2,\dots,m+n-1) &= (p(0,1)(1,2,\dots,m) \bullet_{i-1} q \\ &= p\tau_m \bullet_{i-1} q, \end{aligned}$$

and so the third conditions agree. □

The transposition versions of the second and third cyclic conditions are shown in the following commutative diagrams.

$$\begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_1} & P(m+n-1) \\ \downarrow{\scriptstyle (0,1)\otimes(0,1)} & & \\ P(m)\otimes P(n) & & \downarrow{\scriptstyle (0,1)\phi} \\ \downarrow{\scriptstyle t} & & \\ P(n)\otimes P(m) & \xrightarrow[\bullet_1]{} & P(m+n-1) \end{array} \qquad \begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_i} & P(m+n-1) \\ \downarrow{\scriptstyle (0,1)\otimes id} & & \downarrow{\scriptstyle (0,1)} \\ P(m)\otimes P(n) & \xrightarrow[\bullet_i]{} & P(m+n-1) \end{array} \tag{3.1}$$

Remark 3.2.0.4. Note that the second condition is not really a 'transposition' version since it involves the permutation ϕ. This is an unavoidable effect of the renumbering map, and is illustrative of why in many cases it is useful to work in a non-skeletal setting.

Definition 3.2.0.5 (Category of cyclic skeletal operads)**.** We denote by $Cy\mathcal{O}p_\Sigma$ the category of cyclic skeletal operads. Morphisms are the maps $\theta : P \to Q$ for $P, Q \in Cy\mathcal{O}p_\Sigma$, where $\theta_n : P(n) \to Q(n)$, subject to the same conditions as morphisms in $\mathcal{O}p_{\Sigma_*}$, as well as compatibility with the Σ_{n+1} action by requiring the following diagram to commute.

$$\begin{array}{ccc} P(n) & \xrightarrow{\theta_n} & Q(n) \\ {\scriptstyle (0,1)}\downarrow & & \downarrow{\scriptstyle (0,1)} \\ P(n) & \xrightarrow[\theta_n]{} & Q(n) \end{array}$$

3.3 Non-skeletal cyclic operad definitions

Now we define a non-skeletal cyclic operad, formulated using transpositions analogously to the skeletal one in Proposition 3.2.0.3. Recall that a non-skeletal operad P is a collection $\mathcal{B}ij_*^{op} \to \mathcal{C}$ with composition operations as in Definition 2.3.0.1.

Remark 3.3.0.1. Throughout our discussion of non-skeletal operads, we will alternate between notation x_0 as the basepoint in (X, x_0), and 0 when we talk about bijections that permute the output (which is labelled by the basepoint element). We use 0 as the element to permute, for ease of comparison with the skeletal setting.

Definition 3.3.0.2 (Non-skeletal cyclic operad)**.** A non-skeletal cyclic operad $P \in \mathcal{C}y\mathcal{O}p_{\mathcal{B}ij}$ is an operad $P \in \mathcal{O}p_{\mathcal{B}ij_*}$ together with an extension of the action of basepoint preserving bijections in $\mathcal{B}ij_*$ to all bijections in $\mathcal{B}ij$, generated by the action of the transpositions $(0, x)$ for $x \in X$ viewed as bijections $X \to X$. The compatibility with the composition operation is as follows. For $p \in P(X)$, $q \in P(Y)$, $x' \neq x \in X$, and $1 \in P(\{x\})$ the unit of P,

1. $(1)(0, x) = 1$,
2. $(p \circ_x q)\psi_{x,y} = q(0, y) \circ_y p(0, x)$,
3. $(p \circ_x q)(0, x') = p(0, x') \circ_x q$,

where

$$\begin{aligned} \psi_{x,y} : X \sqcup_x Y &\to Y \sqcup_y X \\ (0,0) &\mapsto (x,1) \\ (y,1) &\mapsto (0,0) \\ (z,0) &\mapsto (z,1) \\ (z,1) &\mapsto (z,0) \end{aligned}$$

for $x \in X, y \in Y, z \in X \cup Y \setminus \{x, y\}$.

In the non-skeletal case, we have the transpositions $(0, x), (0, y)$, and the additional bijection $\psi_{x,y}$ which is needed to achieve the same effect as $(0, 1)\phi$ does in the skeletal case. Note also that we use the indexing notation $(x, 0)$ for $x \in X$ and $(y, 1)$ from the definition of deleted disjoint union for $y \in Y$ in the definition of $\psi_{x,y}$ for clarity. We won't continue to use it though, to avoid confusion with transpositions $(0, x)$ and $(0, y)$.

We have similar commutative diagrams showing the cyclic conditions in this case.

$$\begin{array}{ccc}
P(X) \otimes P(Y) & \xrightarrow{\circ_x} & P(X \sqcup_x Y) \\
\Big\downarrow {\scriptstyle (0,x)\otimes(0,y)} & & \\
P(X) \otimes P(Y) & & \Big\downarrow {\scriptstyle \psi_{x,y}} \\
\Big\downarrow {\scriptstyle t} & & \\
P(Y) \otimes P(X) & \xrightarrow[\circ_y]{} & P(Y \sqcup_y X)
\end{array}
\qquad
\begin{array}{ccc}
P(X) \otimes P(Y) & \xrightarrow{\circ_x} & P(X \sqcup_x Y) \\
\Big\downarrow {\scriptstyle (0,x')\otimes id} & & \Big\downarrow {\scriptstyle (0,x')} \\
P(X) \otimes P(Y) & \xrightarrow[\circ_x]{} & P(X \sqcup_x Y)
\end{array}$$

Definition 3.3.0.3 (Category of cyclic non-skeletal operads)**.** We denote by $Cy\mathcal{O}p_{\mathcal{B}ij}$ the category of cyclic non-skeletal operads. The morphisms are the maps $\theta : P \to Q$ for $P, Q \in Cy\mathcal{O}p_{\mathcal{B}ij}$, where $\theta_X : P(X) \to Q(X)$, subject to the same conditions as $\mathcal{O}p_{\mathcal{B}ij_*}$ as well as the extra compatibility with basepoint permuting bijections by requiring that the diagram below commutes.

$$\begin{array}{ccc}
P(X) & \xrightarrow{\theta_X} & Q(X) \\
\Big\downarrow {\scriptstyle (0,x)} & & \Big\downarrow {\scriptstyle (0,x)} \\
P(X) & \xrightarrow[\theta_X]{} & Q(X)
\end{array}$$

Remark 3.3.0.4. Often, non-skeletal cyclic operads are defined with no distinguished base point, and instead with composition operations ${}_x\circ_y$ for any $x \in X$, $y \in Y$. In [Obr17], this is shown to be equivalent to the definition with composition operations $\circ_x$.

3.4 Equivalence of definitions

Recall from Theorem 2.4.2.4 that we have functors $E^{\#}$ and $R^{\#}$ that give an equivalence of the underlying categories of operads. We will show that we similarly have functors $E^{\#}_{cy}$ and $R^{\#}_{cy}$ between the categories $Cy\mathcal{O}p_{\mathcal{B}ij}$ and $Cy\mathcal{O}p_{\Sigma}$ of cyclic operads, and that these give an equivalence.

Throughout this section we will simplify notation and write $P(0, 1)$ or simply $(0, 1)$ instead of $P((0, 1))$ for the action of the transposition $(0, 1)$.

Proposition 3.4.0.1. *There are functors*

$$E^{\#}_{cy} : Cy\mathcal{O}p_{\mathcal{B}ij} \rightleftarrows Cy\mathcal{O}p_{\Sigma} : R^{\#}_{cy}$$

agreeing with $R^{\#}$ *and* $E^{\#}$ *on the underlying operads.*

Proof. We will check that if an operad is cyclic, the relevant functor indeed gives a cyclic operad in the target category.

$E^{\#}_{cy}$ is well-defined on objects:

Let P be a cyclic $\mathcal{B}ij$-operad. Then $E^{\#}_{cy}P$ is a Σ_*-operad. It remains to check that $E^{\#}_{cy}P$ satisfies the skeletal cyclic conditions. Due to the inclusion $\Sigma_* \hookrightarrow \mathcal{B}ij_*$, we get that for the transposition $(0,1) \in \Sigma_{n+1}$,

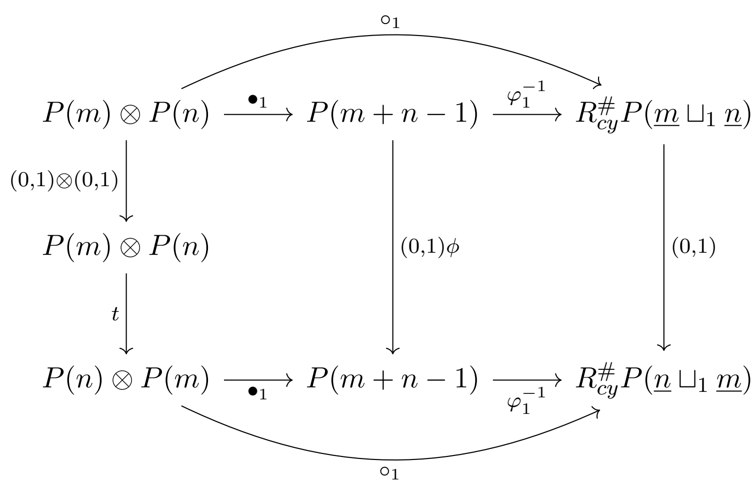
$$E^{\#}_{cy}P(0,1) = P(0,1).$$

Then for $x, y = 1$, the bijection $\varphi_{x,y}$ which maps $0 \mapsto x$ and $y \mapsto 0$ gives the transposition $(0,1)$ composed with the switch map t, and we get the skeletal cyclic conditions from the non-skeletal ones.

$R^{\#}_{cy}$ is well-defined on objects:

(Cyclic condition 2):

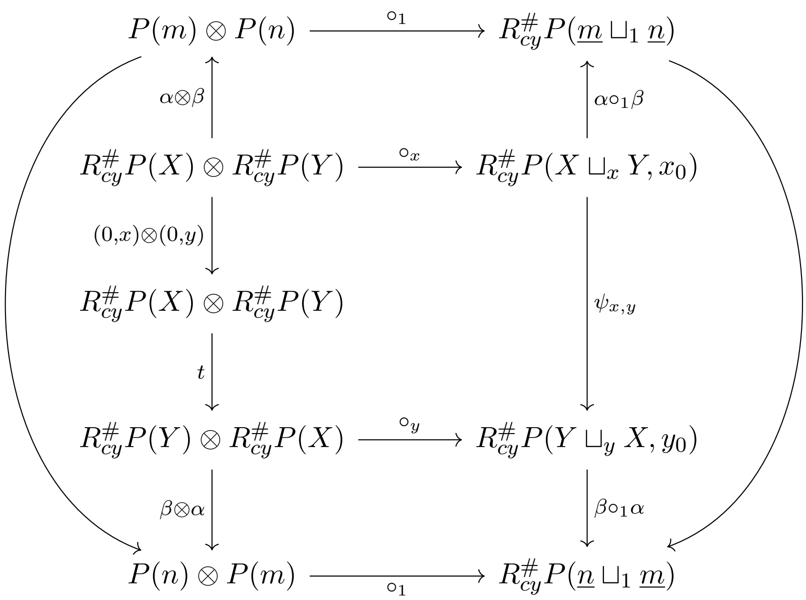
Suppose $P \in Cy\mathcal{O}p_{\Sigma}$. Then the cyclic condition for P is given by the left commutative diagram of (3.1). This can be patched together with a version of diagram (2.1) to give the following commutative diagram.



We also have the commutative diagram below, for maps $\alpha : \underline{m}_* \to (X, x_0), \beta : \underline{n}_* \to (Y, y_0)$ such that $\alpha(1) = x$. This is the same as diagram (2.3) in the non-cyclic case and commutes for the same reason.

$$
\begin{CD}
R^{\#}_{cy}P(X) \otimes R^{\#}_{cy}P(Y) @>{\circ_x}>> R^{\#}_{cy}P(X \sqcup_x Y) \\
@V{\alpha\otimes\beta}VV @VV{\alpha\circ_1\beta}V \\
P(m) \otimes P(n) @>>{\circ_1}> R^{\#}_{cy}P(\underline{m} \sqcup_1 \underline{n})
\end{CD}
\tag{3.2}
$$

By patching the above diagrams together we get a larger diagram. The inner 'square' gives the cyclic condition for $R^{\#}_{cy}P$, which is exactly what we want.



Here we have maps $\alpha : \underline{m}_* \to (X, x_0)$, $\beta : \underline{n}_* \to (Y, y_0)$ such that $\alpha(1) = x$ and $\beta(1) = y$. The top and bottom squares commute as they are equivalent to diagram (3.2). The outer left arrow is given by

$$(\beta \otimes \alpha)t((0, x) \otimes (0, y))(\alpha \otimes \beta)^{-1} = t((0, 1) \otimes (0, 1)),$$

due to the canonical choice of maps α and β and inclusion $\Sigma_* \hookrightarrow \mathcal{B}ij_*$. Similarly, the outer right arrow is given by

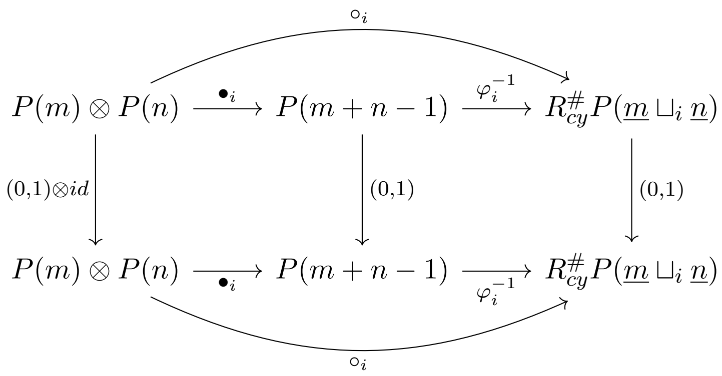
$$(\beta \circ_1 \alpha)\varphi(\alpha \circ_1 \beta)^{-1} = (0, 1),$$

and the left and right 'squares' commute by definition.

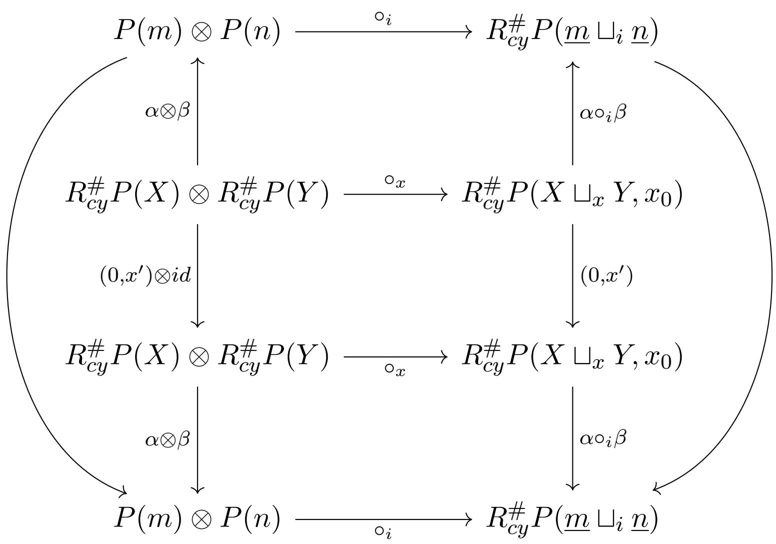
Then the large outer square commutes since P is a cyclic skeletal operad. Therefore, the inner square commutes.

(Cyclic condition 3):

The second cyclic condition for P is given by the right commutative diagram in (3.1). This can also be patched together with a version of diagram (2.1) to give the following commutative diagram.



By again patching with diagram (3.2) we get a similar large diagram for $x \neq x' \in X$, and maps $\alpha, \beta \in \mathcal{B}ij$ such that $\alpha(1) = x$ and $\alpha(i) = x'$.



Then the outer labels are given by

$$(\alpha \otimes \beta)((0, x') \otimes id)(\alpha \otimes \beta)^{-1} = (0, 1) \otimes id,$$
$$(\alpha \circ_i \beta)(0, x')(\alpha \circ_i \beta)^{-1} = (0, 1),$$

and therefore the outer squares commute. Then, the inner square commutes by exactly the same reasoning as for the first cyclic condition.

$E^{\#}_{cy}$ is well-defined on morphisms:

If $\theta : P \to Q$ a map of cyclic $\mathcal{B}ij$ operads, then $E^{\#}(\theta)_n = \theta_{\underline{n}}$ as in the non-cyclic case.

$R^{\#}_{cy}$ is well-defined on morphisms:

This involves equivalent diagrams and checks to the non-cyclic case, and follows from the same reasoning as for objects. □

Theorem 3.4.0.2. *There is an equivalence of categories* $E^{\#}_{cy} : Cy\mathcal{O}p_{\mathcal{B}ij} \to Cy\mathcal{O}p_{\Sigma}$.

Proof. The functor $E^{\#}_{cy}$ is an equivalence of categories, by exactly the same argument as for general operads in Theorem 2.4.2.4 and using Proposition 3.4.0.1. We have that $E^{\#}_{cy}R^{\#}_{cy} = id$, and it follows that $E^{\#}_{cy}$ is full and essentially surjective.

The functor $E^* : Fun(\mathcal{B}ij^{op}, \mathcal{C}) \to Fun(\Sigma^{op}, \mathcal{C})$ is fully faithful. For cyclic operad maps $\theta_1, \theta_2 : P \to Q$ with P and Q objects of $Cy\mathcal{O}p_{\mathcal{B}ij}$, we have that $E^{\#}_{cy}(\theta_1)_n = (\theta_1)_{\underline{n}_*}$ and $E^{\#}_{cy}(\theta_2)_n = (\theta_2)_{\underline{n}_*}$. Therefore, if $E^{\#}_{cy}\theta_1 = E^{\#}_{cy}\theta_2$ then we must have $\theta_1 = \theta_2$.

□

We therefore have the following commutative diagram showing the equivalence of cyclic operad categories, with forgetful functors U' and U to the associated operad categories and functor categories.

$$\begin{array}{ccc}
Cy\mathcal{O}p_{\mathcal{B}ij} & \underset{R^{\#}_{cy}}{\overset{E^{\#}_{cy}}{\rightleftarrows}} & Cy\mathcal{O}p_{\Sigma} \\
{\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U'} \\
\mathcal{O}p_{\mathcal{B}ij_*} & \underset{R^{\#}}{\overset{E^{\#}}{\rightleftarrows}} & \mathcal{O}p_{\Sigma_*} \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
Fun(\mathcal{B}ij^{op}_*, \mathcal{C}) & \underset{R^*}{\overset{E^*}{\rightleftarrows}} & Fun(\Sigma^{op}_*, \mathcal{C})
\end{array}$$

Remark 3.4.0.3. Analogously to Remark 2.4.2.5 one could check the properties of the forgetful functor U', to see that $E^{\#}_{cy}$ is fully faithful and essentially surjective.

3.5 Cyclic non-symmetric operads

We can have both cyclic symmetric operads, such as $\mathcal{A}ss_{\Sigma}$, and cyclic non-symmetric operads, such as $\mathcal{A}ss$. In the case of the latter, this means that we do not have the usual action of Σ_n on $P(n)$, but we do have a cyclic action of C_{n+1} on $P(n)$. Therefore, the extra action is not an extension of the action of a smaller group.

Definition 3.5.0.1 (Cyclic non-symmetric operad)**.** A non-symmetric cyclic operad is a non-symmetric operad P with an action of the cyclic group C_{n+1} on $P(n) \in P$, such that for $p \in P(m), q \in P(n)$, the following compatibility conditions hold:

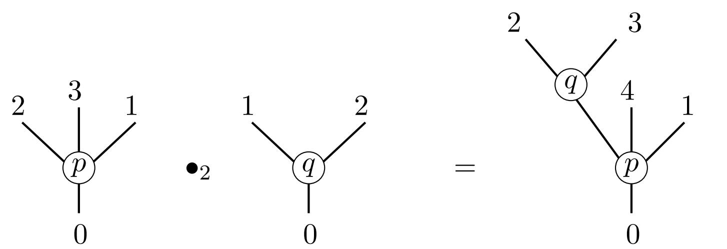
1. $(1)\tau_1 = 1$,
2. $(p \bullet_1 q)\tau_{m+n-1} = (q\tau_n) \bullet_n (p\tau_m)$,
3. $(p \bullet_i q)\tau_{m+n-1} = (p\tau_m) \bullet_{i-1} (q\tau_n)$.

This definition appears exactly the same as the symmetric version, only in this case the cycles τ_k are generators for the cyclic group action of C_{k+1}, which is not the extension of a Σ_k action.

Note that this also means one cannot write a version of this definition using transpositions in this case.

Example 3.5.0.2 (Cyclically labelled trees)**.** The collection of all cyclically labelled trees gives a non-symmetric cyclic operad as described in [RW02]. There is no symmetric group action as this would not preserve the cyclic order of the labels, however one could create a symmetric cyclic operad by taking the product with the symmetric group.

We require that the composition of two cyclically labelled trees is itself a cyclically labelled tree in order for this to give a well-defined operad. This is clearly true, as the renumbering map just block shifts labels, while preserving the order of leaf labels within each tree. The following diagram is an example of this.



$p \in P(3)$ $\quad$ $q \in P(2)$ $\quad$ $p \bullet_2 q \in P(4)$

There is an obvious action of the cyclic group C_{n+1} on $P(n)$ that permutes the labels including the root cyclically. This action on any given tree gives another cyclically labelled tree. It is easy to check that the cyclic conditions hold.

They correspond to Stasheff polyhedra first introduced in [Sta61], and therefore give an A_∞-operad.

3.6 Uniqueness of the cyclic structure

A natural question to ask is whether or when the cyclic structure satisfied by a cyclic operad is unique. As we have seen, an action of the symmetric group gives a group representation, and if an action of Σ_{n+1} restricts to a Σ_n action, then the associated Σ_{n+1} group representation restricts to the Σ_n representation. For a given Σ_n representation, there are often multiple Σ_{n+1} representations that restrict to it. This means there is no representation theoretic reason why the cyclic structure on a cyclic operad should be unique.

Example 3.6.0.1 (An operad with two cyclic structures [HRY19, Example 8.9])**.** An example of an operad admitting two distinct cyclic structures is given by Hackney, Robertson and Yau. For the group

$$\mathbb{Z}/2 \times \mathbb{Z}/2 = \{0,1\} \times \{0,1\},$$

we have an operad in *Set* given by

$$P(n) := \begin{cases} \{(0,0),(0,1),(1,0),(1,1)\} & \text{if } n = 1, \\ \emptyset & \text{if } n \neq 1, \end{cases}$$

together with the group operation.

Then we have a cyclic action of Σ_2 on $P(1)$ by transposing the elements $(0,1)$ and $(1,0)$. We also have the action that fixes all the elements. These two cyclic structures are not isomorphic.

This is an example of a general result for an abelian group A. With the group elements in arity 1 as above, we have the operad P_A given by

$$P_A(n) = \begin{cases} A & \text{if } n = 1, \\ \emptyset & \text{if } n \neq 1. \end{cases}$$

Composition is given by the group operation

$$P_A(1) \otimes P_A(1) \xrightarrow{\bullet_1} P_A(1)$$
$$A \times A \mapsto A.$$

This satisfies associativity as the group operation is associative, and equivariance is satisfied as we have trivial Σ_1 action at arity 1.

Then the cyclic structures on P_A are in bijection with order 1 and 2 automorphisms of A.

3.7 Examples

We will mention a number of interesting examples, as well as non-examples where any cyclic structure isn't compatible with the underlying operad structure. In particular, we will look at the associative operad and the Lie operad in more detail. There are a number of examples of cyclic operads given by Getzler and Kapranov in [GK95] and [Get95].

Example 3.7.0.1 (The framed little disks operad)**.** The framed little disks operad is a cyclic operad. The proof of this is given by Budney in [Bud08].

We now focus on a key example that we defined in Chapter 2. The associative operad can be defined as either a symmetric or non-symmetric operad. It is a quadratic operad and in both the symmetric and non-symmetric case, it is cyclic. Here we work in the category $(\mathrm{Mod}_k, \otimes, k)$, where k is a field.

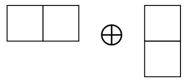
Example 3.7.0.2 (The symmetric associative operad)**.** We describe a cyclic structure on $\mathcal{A}ss_\Sigma$ by first writing the action of Σ_3 on $\mathcal{A}ss_\Sigma(2)$. The structure for $n > 2$ is determined by the composition maps and compatibility conditions.

First note that $\mathcal{A}ss_\Sigma(1) = k\{Id\}$. The cyclic Σ_2 action on this is the one giving the trivial representation. This has no implications for higher actions since composing any $p \in \mathcal{A}ss_\Sigma(k)$ with an element of $\mathcal{A}ss_\Sigma(1)$ gives us back an element of $\mathcal{A}ss_\Sigma(k)$.

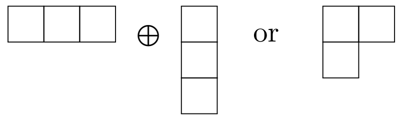
We have that

$$\mathcal{A}ss(2) = k\Sigma_2[\mu] = k\{\mu, \mu(12)\}.$$

The Σ_2 action on $\mathcal{A}ss(2)$ has the regular representation, with Young diagrams as below.



Then we have the following two possible Σ_3 representations that restrict to the regular representation of Σ_2.



The first is the regular representation, and the second is the standard (or defining) representation of Σ_3. Both are two-dimensional representations.

If we choose the regular representation to describe the extra action of Σ_3 on $\mathcal{A}ss(2)$, then this gives us the action associated with the representation $Ind_{C_{n+1}}^{\Sigma_{n+1}}1$ of Σ_{n+1} on $\mathcal{A}ss(n)$, which does indeed restrict to the regular representation of Σ_n. This is because for a subgroup $H < G$, we have a kG action on the induced representation

$$Ind_H^G V = V \otimes_{kH} kG.$$

Therefore, we have

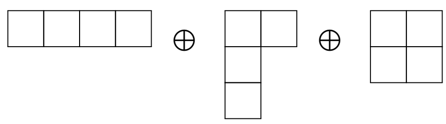
$$Ind_{C_{n+1}}^{\Sigma_{n+1}}1 = k \otimes_{kC_{n+1}} k\Sigma_{n+1}$$

with the basis any set of coset representatives for C_{n+1} in Σ_{n+1}. We can choose $\{\pi | \pi \in \Sigma_n\}$, so it restricts to the regular representation of Σ_n.

We will see what this looks like explicitly for $n = 3$. We have

$$\mathcal{A}ss(3) = k\Sigma_3[\mu(1 \otimes \mu)] = k\Sigma_3[\mu(\mu \otimes 1)],$$

where the second equality is simply associativity. As usual, Σ_3 acts on $\mathcal{A}ss(3)$ with the regular representation. Then we get the Σ_4 representation $S^{(4)} \oplus S^{(2,1,1)} \oplus S^{(2,2)}$ with Young diagrams below.



This can be shown by character calculations.

We will show by direct calculation that the other possible representation of Σ_3 on $\mathcal{A}ss_\Sigma(2)$ is not possible, and that therefore the cyclic structure is unique in this example.

Proposition 3.7.0.3. *The cyclic structure on $\mathcal{A}ss_\Sigma$ generated by the regular representation of Σ_3 on $\mathcal{A}ss_\Sigma(2)$ is unique.*

Proof. The only alternative Σ_3 action on the level $\mathcal{A}ss_\Sigma(2)$ is the one that gives the standard representation of Σ_3. The standard representation is given by the action of Σ_3 on the two-dimensional vector space

$$\{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$$

with basis $\{x_3 - x_1, x_3 - x_2\}$. The permutation of indices by Σ_3 give an action on the vector space.

Then, if we make the allocation

$$\mu = x_0 - x_1$$
$$\mu(12) = x_0 - x_2,$$

we have that $(1,2)$ switches the two operations as required. Then we consider the action of the extra transposition $(0,1)$. We have

$$\mu(0,1) = x_1 - x_0 = -\mu$$
$$\mu(1,2)(0,1) = x_1 - x_2 = \mu(1,2) - \mu.$$

We will check if the associativity condition is compatible with this action. Recall from the example in Section 2.6.0.5 that this means

$$\mu(\mu \otimes id) = \mu(id \otimes \mu).$$

This holds for multiplication operations in $\mathcal{A}ss_\Sigma(2)$, so for both μ and $\mu(1,2)$. We need this to be compatible with the extra action of Σ_3, and so require that it also holds for $\mu(0,1)$ and $\mu(1,2)(0,1)$.

Indeed, we have that

$$\mu(0,1)(\mu(0,1) \otimes id) = \mu(0,1)(id \otimes \mu(0,1)).$$

For the action of $(0,1)$ on $\mu(1,2)$ we need that

$$((\mu(1,2))(0,1) \otimes 1)(\mu(1,2))(0,1) = (1 \otimes (\mu(1,2))(0,1))(\mu(1,2))(0,1).$$

The left-hand side gives

$$\begin{aligned}
&(\mu(1,2) - \mu)(id \otimes (\mu(1,2) - \mu)) \\
=&(\mu(1,2) - \mu)(id \otimes \mu(1,2) - id \otimes \mu) \\
=&\mu(1,2)(id \otimes \mu(1,2)) - \mu(id \otimes \mu(1,2)) - \mu(1,2)(id \otimes \mu) + \mu(id \otimes \mu).
\end{aligned}$$

On the right-hand side, we have

$$\begin{aligned}
&(\mu(1,2) - \mu)((\mu(1,2) - \mu) \otimes id) \\
=&(\mu(1,2) - \mu)(\mu(1,2) \otimes id - \mu \otimes id) \\
=&\mu(1,2)(\mu(1,2) \otimes id) - \mu(\mu(1,2) \otimes id) - \mu(1,2)(\mu \otimes id) + \mu(id \otimes \mu).
\end{aligned}$$

We have that

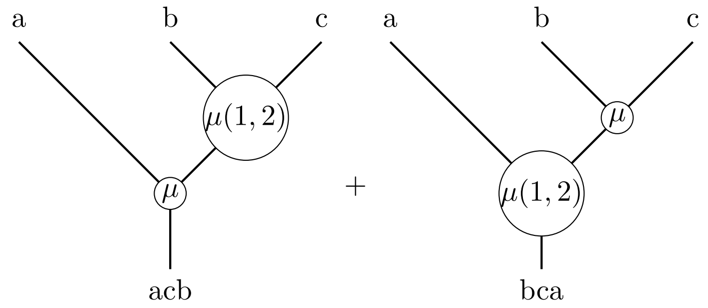
$$\mu(1,2)(id \otimes \mu(1,2) = \mu(1,2)(\mu(1,2) \otimes id)$$

by definition of $\mathcal{A}ss_\Sigma$. So it remains to check whether the expressions

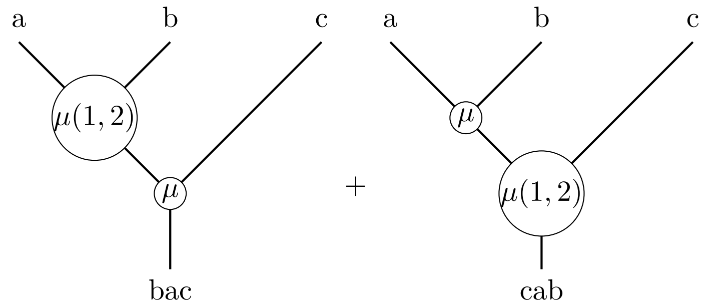
$$\mu(id \otimes \mu(1,2)) + \mu(1,2)(id \otimes \mu) \quad \text{and} \quad \mu(\mu(1,2) \otimes id) + \mu(1,2)(\mu \otimes id)$$

are equal or not.

The left can be depicted with trees as follows.



For the right-hand side, we have



Then we see that these are not equal, and therefore this does not give a cyclic structure on $\mathcal{A}ss_\Sigma$. □

Example 3.7.0.4 (The non-symmetric associative operad)**.** Recall the non-symmetric associative operad $\mathcal{A}ss$ from Example 2.6.0.6. This is a non-symmetric cyclic operad, which means we do not have a Σ_n action on $\mathcal{A}ss(n)$, only a C_{n+1} action.

Example 3.7.0.5 (The Lie operad)**.** Recall the algebraic Lie operad from Example 2.6.0.7. This is also a cyclic operad, and Kontsevich gives a Σ_{n+1} action on $\mathcal{L}ie(n)$ in [Kon93] that gives a cyclic structure.

3.7.1 Examples of non-cyclic operads

We have seen that many fundamental and well-known operads are indeed cyclic. Below we give some examples of non-cyclic operads.

Example 3.7.1.1 (The rooted tree operad)**.** The rooted tree operad, sometimes denoted by $\mathcal{R}Tree$ as in [MSS02], is the operad with elements rooted n-trees. Composition is given by grafting of n-trees where the root of the second tree is grafted into the i^{th} leaf of the first tree for the composition $\bullet_i$. Then Σ_n permutes the labels of the leaves of the tree, and $\mathcal{R}Tree$ is not cyclic since we cannot mix up the root with the other leaves.

Example 3.7.1.2 (The little disks operad)**.** The (unframed) little disks operad is not cyclic. This can be shown by the fact its homology is not cyclic, as shown in [GK95, Proposition 3.18].

Example 3.7.1.3 (The braid operad)**.** The Braid operad is another example described by Getzler and Kapranov [GK95, p. 3.17] that is not cyclic.

3.8 Anticyclic operads

Finally, we will briefly discuss a variant of cyclic structure that will be useful to us later. This modification of cyclic structure on an operad, is called anticyclic structure, which is essentially the property that an operad is cyclic up to a twist by sign. Getzler and Kapranov introduced anticyclic operads, and they are of particular importance when taking the operadic suspension or desuspension of a cyclic operad, which we defined in Section 2.9.

Definition 3.8.0.1 (Anticyclic operad, [MSS02, Definition 5.9])**.** A symmetric operad P is anticyclic if the following conditions hold for $1 \in P(1)$ the unit of the operad P, $p \in P(m)$, $q \in P(n)$, $2 \leq i \leq m$, and τ_k the cycle $(0, 1, ..., k) \in \Sigma_{k+1}$,

1. $(1)\tau_1 = -1$,
2. $(p \bullet_1 q)\tau_{m+n-1} = -(q\tau_n) \bullet_n (p\tau_m)$,
3. $(p \bullet_i q)\tau_{m+n-1} = (p\tau_m) \bullet_{i-1} q$.

Note that this means that the commutative diagrams corresponding to cyclic conditions 1 and 2 commute up to a sign of -1, and the diagram for condition 3 commutes as in the cyclic case.

Then as described by Getzler and Kapranov, the resulting operad after suspension of a cyclic operad is not cyclic, but anticyclic.

Chapter 4

Cooperads

Cooperads are a dual notion to operads, used to study coalgebras in the same way operads encode algebras. In many of these cases, it is simpler to work directly with cooperad structures, rather than operads.

In this chapter we will define skeletal and non-skeletal cooperads, and prove the equivalence of these definitions in Theorem 4.4.0.1. As this is essentially dual to the procedure we have seen for operads, we will omit a lot of the detail here.

Then, we will define cyclic cooperads and again show the equivalence of the skeletal and non-skeletal definitions in Theorem 4.8.2.1. The explicit cyclic cooperad descriptions are of particular importance to us, because in Chapter 6 we will give an explicit construction of a cooperad in this way. It will be useful to compare directly with the diagrams in these definitions when checking that the necessary conditions are satisfied.

4.1 Notation for cocomposition maps

In the operad setting, we had skeletal composition maps $\bullet_i$ and non-skeletal composition maps $\circ_x$, where in the skeletal setting the renumbering map φ_i defined in Definition 2.1.1.1 was built in. We have the same in the cooperad case, and we will use similar notation to differentiate between the two settings and for easy comparison with the operad case. We define the skeletal and non-skeletal cocomposition operations respectively. For (X, x_0), $(Y, y_0) \in \mathcal{B}ij_*$, $x \in X$, and $i \in \underline{m}$, these are written as follows

$$\overline{\bullet}_i : P(m+n-1) \to P(m) \otimes P(n),$$
$$\overline{\circ}_x : P(X \sqcup_x Y, x_0) \to P(X, x_0) \otimes P(Y, y_0).$$

Note that we still use the notation $\bullet_i$ and $\circ_x$, as in Section 2.1.2, to write compositions of permutations and bijections that act on the larger sets.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category.

4.2 Skeletal cooperads

Definition 4.2.0.1 (Skeletal cooperad)**.** A skeletal cooperad in $\mathcal{C}$ is a collection $P : \Sigma_*^{op} \to \mathcal{C}$ along with cocomposition operations

$$\bar{\bullet}_i : P(m+n-1) \to P(m) \otimes P(n),$$

for all $1 \leq i \leq m$, such that coassociativity, equivariance and counit axioms are all satisfied.

The commutative diagrams for the equivariance and coassociativity are as follows.

- (Equivariance)

$$\begin{array}{ccc} P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_{\sigma(i)}} & P(m+n-1) \\ {\scriptstyle \sigma\otimes\tau}\downarrow & & \downarrow{\scriptstyle \sigma\circ_i\tau} \\ P(m) \otimes P(n) & \xleftarrow[\bar{\bullet}_i]{} & P(m+n-1) \end{array}$$

- (Coassociativity) There are two cases. Firstly, let $1 \leq i \leq l$ and $i \leq j \leq i+m-1$. Then if we cocompose once at an input to $p \in P(l)$, and once at an input of $q \in P(m)$, we have

$$\begin{array}{ccc} P(l) \otimes P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_i \otimes id} & P(l+m-1) \otimes P(n) \\ {\scriptstyle id\otimes\bar{\bullet}_j}\uparrow & & \uparrow{\scriptstyle \bar{\bullet}_{j+i-1}} \\ P(l) \otimes P(m+n-1) & \xleftarrow[\bar{\bullet}_i]{} & P(l+m+n-2) \end{array}$$

Secondly, let $1 \leq i \leq l$ and $j < i$. That is, we cocompose twice at inputs to $p \in P(l)$. Then we have

$$\begin{array}{ccc} P(l) \otimes P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_i \otimes id} & P(l+m-1) \otimes P(n) \\ {\scriptstyle id\otimes t}\uparrow & & \\ P(l) \otimes P(n) \otimes P(m) & & \uparrow{\scriptstyle \bar{\bullet}_j} \\ {\scriptstyle \bar{\bullet}_j\otimes id}\uparrow & & \\ P(l+n-1) \otimes P(m) & \xleftarrow[\bar{\bullet}_i]{} & P(l+m+n-2). \end{array}$$

If $j > i$, we have the diagram below.

$$\begin{array}{ccc} P(l) \otimes P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_i \otimes id} & P(l+m-1) \otimes P(n) \\ {\scriptstyle id\otimes t}\uparrow & & \\ P(l) \otimes P(n) \otimes P(m) & & \uparrow{\scriptstyle \bar{\bullet}_{j+m-1}} \\ {\scriptstyle \bar{\bullet}_j\otimes id}\uparrow & & \\ P(l+n-1) \otimes P(m) & \xleftarrow[\bar{\bullet}_i]{} & P(l+m+n-2) \end{array}$$

- (Counit) Let $\mathbf{1} \in \mathcal{C}$ be the counit object of the underlying category. Then there is a map $\overline{\eta} : P(1) \to \mathbf{1}$ that interacts with the composition operation such that the following diagrams commute for all $i \in \underline{n}$.

$$\begin{array}{ccc} P(n) \otimes \mathbf{1} & \xleftarrow{id \otimes \eta} & P(n) \otimes P(1) \\ & \nwarrow & \uparrow \bar{\bullet}_1 \\ & & P(n) \end{array} \qquad \begin{array}{ccc} \mathbf{1} \otimes P(n) & \xleftarrow{\eta \otimes id} & P(1) \otimes P(n) \\ & \nwarrow & \uparrow \bar{\bullet}_i \\ & & P(n) \end{array}$$

As with skeletal operads, the renumbering map is built into the cocomposition operation in order for the resulting operations to be Σ_*-operads. We have that

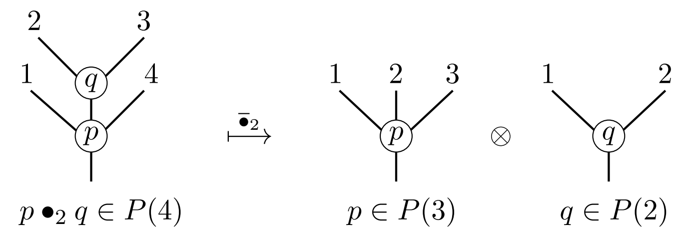
$$\bar{\bullet}_i = \bar{\circ}_i \varphi_i^{-1}.$$

Remark 4.2.0.2. As in the operad case, we are using partial cocomposition operations here. One can also define cooperads with a general cocomposition

$$P(n) \to P(k) \otimes P(i_1) \otimes \cdots \otimes P(i_k),$$

analogously to the composition in Remark 2.2.0.3.

Remark 4.2.0.3. Where we could think of operations in an operad as trees and composition as grafting trees, we can similarly think of cocomposition in a cooperad as the 'ungrafting' of trees. Below is a picture with trees of the cocomposition operation.



Remark 4.2.0.4. Recall from Remark 2.2.0.4 that an operad may be viewed as a monoid in the category of symmetric sequences with composition operations. An alternative approach is to define a cooperad in a symmetric monoidal category $\mathcal{C}$ as a comonoid in the category of symmetric sequences in $\mathcal{C}$ with cocomposition operations.

However, there are some complications in the comonoid case. A cooperad can also be viewed as an operad in the category $\mathcal{C}^{op}$, and therefore a monoid in the category of symmetric sequences in $\mathcal{C}^{op}$ with composition. The composition operation is the monoidal product in the category of symmetric sequences, and in order for this to commute with colimits, we require that the symmetric monoidal structure on $\mathcal{C}$ is closed. In the case where this is true for a category $\mathcal{C}$, it often isn't the case for the dual category $\mathcal{C}^{op}$. This is discussed further by Ching in [Chi12a] and [Chi05].

The collection of skeletal cooperads and cooperad maps forms a category.

Definition 4.2.0.5 (Category of skeletal cooperads)**.** We denote by $\mathcal{C}oop_{\Sigma_*}$ the category of skeletal cooperads and cooperad maps. The maps must commute with the cocomposition operation, the action of permutations, and counit maps. These conditions are given by commutative diagrams. For the compatibility with cocomposition we have

$$\begin{array}{ccc} P(m) \otimes P(n) & \xrightarrow{\theta_m \otimes \theta_n} & Q(m) \otimes Q(n) \\ \uparrow{\scriptstyle \bar{\bullet}_i} & & \uparrow{\scriptstyle \bar{\bullet}_i} \\ P(m+n-1) & \xrightarrow[\theta_{m+n-1}]{} & Q(m+n-1). \end{array}$$

The other diagrams correspond exactly to those in Definition 2.2.0.5 for the operad case, so we omit them here.

4.3 Non-skeletal cooperads

Now we give the definition of a non-skeletal cooperad.

Definition 4.3.0.1 (Non-skeletal cooperad)**.** A non-skeletal cooperad in $\mathcal{C}$ is a collection $P : \mathcal{B}ij_*^{op} \to \mathcal{C}$ along with cocomposition operations

$$\bar{\circ}_x : P(X \sqcup_x Y, x_0) \to P(X, x_0) \otimes P(Y, y_0),$$

for all (X, x_0), $(Y, y_0) \in \mathcal{B}ij_*$, such that coassociativity, equivariance and counit axioms are satisfied.

The equivariance and coassociativity axioms are given by the following commutative diagrams.

- (Equivariance)

$$\begin{array}{ccc} P(X', x_0') \otimes P(Y', y_0') & \xleftarrow{\bar{\circ}_{\sigma(x)}} & P(X' \sqcup_{\sigma(x)} Y', x_0') \\ \downarrow{\scriptstyle \sigma \otimes \tau} & & \downarrow{\scriptstyle \sigma \circ_x \tau} \\ P(X, x_0) \otimes P(Y, y_0) & \xleftarrow[\bar{\circ}_x]{} & P(X \sqcup_x Y, x_0) \end{array}$$

- (Coassociativity) There are two cases. Firstly, if we can write (W, w_0) as the union $(X \sqcup_x Y \sqcup_y Z, x_0)$ with $x \in X, y \in Y$, we have the diagram below.

$$\begin{array}{ccc} P(X, x_0) \otimes P(Y, y_0) \otimes P(Z, z_0) & \xleftarrow{\bar{\circ}_x \otimes id} & P(X \sqcup_x Y, x_0) \otimes P(Z, z_0) \\ \uparrow{\scriptstyle id \otimes \bar{\circ}_y} & & \uparrow{\scriptstyle \bar{\circ}_y} \\ P(X, x_0) \otimes P(Y \sqcup_y Z, y_0) & \xleftarrow[\bar{\circ}_x]{} & P(X \sqcup_x Y \sqcup_y Z, x_0) = P(W, w_0) \end{array}$$

  If we can write (W, w_0) as the union $(X \sqcup_x Y \sqcup_{x'} Z, x_0)$ with $x, x' \in X$, we have the following diagram.

$$\begin{array}{ccc} P(X,x_0)\otimes P(Y,y_0)\otimes P(Z,z_0) & \xleftarrow{\overline{\circ}_x\otimes id} & P(X\sqcup_x Y,x_0)\otimes P(Z,z_0)\\ \uparrow{\scriptstyle id\otimes t} & & \\ P(X,x_0)\otimes P(Z,z_0)\otimes P(Y,y_0) & & \uparrow{\scriptstyle \overline{\circ}_y}\\ \uparrow{\scriptstyle \overline{\circ}_y\otimes id} & & \\ P(X\sqcup_{x'} Z,x_0)\otimes P(Y,y_0) & \xleftarrow[\overline{\circ}_x]{} & P(X\sqcup_x Y\sqcup_{x'} Z,x_0)=P(W,w_0)\end{array}$$

- (Counit) There exists a map $\overline{\eta}: P(W,w_0)\to \mathbf{1}$ where $(W,w_0)=\{w,w_0\}$, the set with one non-basepoint element such that the following diagrams commute for any $x\in X$.

$$\begin{array}{ccc} P(X,x_0)\otimes \mathbf{1} & \xleftarrow{id\otimes\eta} & P(X,x_0)\otimes P(W,w_0)\\ & \nwarrow & \uparrow{\scriptstyle \overline{\circ}_w}\\ & & P(X,x_0)\end{array} \qquad \begin{array}{ccc} \mathbf{1}\otimes P(X,x_0) & \xleftarrow{\eta\otimes id} & P(W,w_0)\otimes P(X,x_0)\\ & \nwarrow & \uparrow{\scriptstyle \overline{\circ}_x}\\ & & P(X,x_0)\end{array}$$

We have a category of non-skeletal cooperads and cooperad maps.

Definition 4.3.0.2 (Category of non-skeletal cooperads)**.** We denote by $\mathcal{C}oop_{\mathcal{B}ij_*}$ the category of non-skeletal cooperads and cooperad maps. The maps must commute with the cocomposition operation, with the action of bijections, and with counit maps. These conditions are given by commutative diagrams. For the compatibility with cocomposition we have

$$\begin{array}{ccc} P(X,x_0)\otimes P(Y,y_0) & \xrightarrow{\theta_X\otimes\theta_Y} & Q(X,x_0)\otimes Q(Y,y_0)\\ \uparrow{\scriptstyle \overline{\circ}_x} & & \uparrow{\scriptstyle \overline{\circ}_x}\\ P(X\sqcup_x Y,x_0) & \xrightarrow[\theta_{X\sqcup_x Y}]{} & Q(X\sqcup_x Y,x_0).\end{array}$$

The other diagrams correspond exactly to those in Definition 2.3.0.2 for the operad case, so we omit them here.

4.4 Equivalence of definitions

The functors E^*, R^* give rise to functors

$$E^{\#}_{co}:\mathcal{C}oop_{\Sigma_*} \rightleftarrows \mathcal{C}oop_{\mathcal{B}ij_*}:R^{\#}_{co}$$

on the cooperad categories.

Theorem 4.4.0.1. *There is an equivalence of categories $E^{\#}_{co}:\mathcal{C}oop_{\mathcal{B}ij_*}\to\mathcal{C}oop_{\Sigma_*}$.*

Proof. All the diagrams in the proof of Theorem 2.4.2.4 with the composition arrows replaced by cocomposition and their direction swapped apply in this proof. Therefore, we can simply dualise the argument to get that the functors $E^{\#}_{co}$ and $R^{\#}_{co}$ are well-defined on the cooperad categories $\mathcal{C}oop_{\Sigma_*}$ and $\mathcal{C}oop_{\mathcal{B}ij_*}$, and that $E^{\#}_{co}R^{\#}_{co} = id$. The functor $E^{\#}_{co}$ is essentially surjective and fully faithful, therefore giving the equivalence. □

Remark 4.4.0.2. We can also obtain an equivalence of skeletal and non-skeletal cooperad definitions in a category $\mathcal{C}$ with the necessary properties since we have that a cooperad is an operad in $\mathcal{C}^{op}$ and using the equivalence of operad categories in Theorem 2.4.2.4. One would first need to show that the category of skeletal (or non-skeletal) cooperads in $\mathcal{C}$ is equivalent to category of skeletal (or non-skeletal) operads in $\mathcal{C}^{op}$.

We therefore have the following commutative diagram showing the equivalence of cooperad categories, with forgetful functors V to the associated functor categories.

$$\begin{array}{ccc} \mathcal{C}oop_{\mathcal{B}ij_*} & \underset{R^{\#}_{co}}{\overset{E^{\#}_{co}}{\rightleftarrows}} & \mathcal{C}oop_{\Sigma_*} \\ V\big\downarrow & & \big\downarrow V \\ Fun(\mathcal{B}ij_*^{op},\mathcal{C}) & \underset{R^*}{\overset{E^*}{\rightleftarrows}} & Fun(\Sigma_*^{op},\mathcal{C}) \end{array} \tag{4.1}$$

4.5 Cooperad coalgebras

As with operads, we can define a coalgebra over a cooperad similarly. The definition below can be found in [LV12, p. 5.7.3].

A coalgebra C over a cooperad P is a vector space C and a map

$$\Delta_C : C \to \hat{P} := \prod_n (P(n) \otimes C^{\otimes n}),$$

such that the following diagrams commute.

$$\begin{array}{ccccc} (P \circ P)(C) & = & \hat{P}(\hat{P}(C)) & \xleftarrow{\hat{P}(\Delta_C)} & \hat{P}(C) \\ \Delta(C)\big\uparrow & & & & \big\uparrow\Delta_C \\ \hat{P}(C) & & \xleftarrow[\Delta_C]{\qquad\qquad} & & C \end{array} \qquad \begin{array}{ccc} I(C) & \xleftarrow{\eta(C)} & \hat{P}(C) \\ & \nwarrow^{\simeq} & \big\uparrow\Delta_C \\ & & C \end{array}$$

Remark 4.5.0.1. One can also define an algebra over a cooperad.

4.6 Examples

- The coassociative cooperad is the cooperad whose coalgebras are coassociative coalgebras.
- The cocommutative cooperad is the cooperad whose coalgebras are cocommutative coalgebras.

4.7 Cooperad (co)homology

Analogously to operad homology, the homology of a topological cooperad gives a cooperad in Mod_k. The cohomology of a cooperad in based spaces gives an operad in Mod_k.

For cooperads, we need in both cases that the (co)homology gives flat k-modules, and for the cohomology, we need that $H^*(P)$ is finitely generated. This is discussed by Ching in [Chi05, p. 9.1].

4.8 Cyclic cooperads

As we did in Chapter 3, we extend in the same way the proof of equivalence of the skeletal and non-skeletal definitions to the cyclic case. Much of this dual, and the equivalence is a straightforward consequence of the result for cyclic operads.

4.8.1 Cyclic cooperad definitions

First, recall the map t given by (1.1) in Section 3.1.2 that swaps the order of two operations. We also recall from Definition 3.2.0.1 the notation $\tau_n := (0, 1, \ldots, n)$ for the cyclic permutation of $n+1$ elements.

Definition 4.8.1.1 (Skeletal cyclic cooperad)**.** A skeletal cyclic cooperad P is a skeletal cooperad with an action of Σ_{n+1} on $P(n)$ such that $(1)\tau_1 = 1$ for $1 \in P(1)$ the counit of P, and such that the following diagrams commute.

$$
\begin{array}{ccc}
P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_n} & P(m+n-1) \\
\downarrow{\scriptstyle \tau_m \otimes \tau_n} & & \\
P(m) \otimes P(n) & & \downarrow{\scriptstyle \tau_{m+n-1}} \\
\downarrow{\scriptstyle t} & & \\
P(n) \otimes P(m) & \xleftarrow[\bar{\bullet}_1]{} & P(m+n-1)
\end{array}
\qquad
\begin{array}{ccc}
P(m) \otimes P(n) & \xleftarrow{\bar{\bullet}_i} & P(m+n-1) \\
\downarrow{\scriptstyle \tau_m \otimes id} & & \downarrow{\scriptstyle \tau_{m+n-1}} \\
P(m) \otimes P(n) & \xleftarrow[\bar{\bullet}_{i-1}]{} & P(m+n-1)
\end{array}
\tag{4.2}
$$

Proposition 4.8.1.2. *Definition 4.8.1.1 can equivalently be formulated with the condition that the following diagrams commute, instead of the diagrams in (4.2). That is, with the extra action of* Σ_{n+1} *generated by the transposition* $(0, 1)$ *rather than by cycles. The counit condition is the same.*

$$
\begin{array}{ccc}
P(m)\otimes P(n) & \xleftarrow{\bar{\bullet}_1} & P(m+n-1) \\
\downarrow{\scriptstyle (0,1)\otimes(0,1)} & & \\
P(m)\otimes P(n) & & \downarrow{\scriptstyle (0,1)\phi} \\
\downarrow{\scriptstyle t} & & \\
P(n)\otimes P(m) & \xleftarrow[\bar{\bullet}_1]{} & P(m+n-1)
\end{array}
\qquad
\begin{array}{ccc}
P(m)\otimes P(n) & \xleftarrow{\bar{\bullet}_1} & P(m+n-1) \\
\downarrow{\scriptstyle (0,1)\otimes id} & & \downarrow{\scriptstyle (0,1)} \\
P(m)\otimes P(n) & \xleftarrow[\bar{\bullet}_1]{} & P(m+n-1)
\end{array}
$$

Recall that ϕ is the permutation from Proposition 3.2.0.3,

$$
\begin{pmatrix}
1 & 2 & 3 & \dots & m & m+1 & \dots & m+n-1 \\
1 & n+1 & n+2 & \dots & m+n-1 & 2 & \dots & n
\end{pmatrix}.
$$

Definition 4.8.1.3 (Category of skeletal cyclic cooperads)**.** We denote by $Cy\mathcal{C}oop_{\Sigma}$ the category of skeletal cyclic cooperads and cooperad maps that commute with the actions of permutations, cocomposition and counits.

We now give the non-skeletal definition, before moving on to the proof of the equivalence of the two definitions.

Definition 4.8.1.4 (Non-skeletal cyclic cooperad)**.** A non-skeletal cyclic cooperad P is a non-skeletal cooperad together with an extension of the action of basepoint preserving bijections in $\mathcal{B}ij_*$ to all bijections in $\mathcal{B}ij$, generated by the action of the transpositions $(0,x)$ for $x \in X$ viewed as bijections $X \to X$, such that $(1)(0,x) = 1$ for $1 \in P(\{x\})$ the unit of P, and such that the following diagrams commute.

$$
\begin{array}{ccc}
P(X)\otimes P(Y) & \xleftarrow{\bar{\circ}_x} & P(X\sqcup_x Y) \\
\downarrow{\scriptstyle (0,x)\otimes(0,y)} & & \\
P(X)\otimes P(Y) & & \downarrow{\scriptstyle \psi_{x,y}} \\
\downarrow{\scriptstyle t} & & \\
P(Y)\otimes P(X) & \xleftarrow[\bar{\circ}_y]{} & P(Y\sqcup_y X)
\end{array}
\qquad
\begin{array}{ccc}
P(X)\otimes P(Y) & \xleftarrow{\bar{\circ}_x} & P(X\sqcup_x Y) \\
\downarrow{\scriptstyle (0,x')\otimes id} & & \downarrow{\scriptstyle (0,x')} \\
P(X)\otimes P(Y) & \xleftarrow[\bar{\circ}_y]{} & P(X\sqcup_x Y)
\end{array}
\tag{4.3}
$$

Definition 4.8.1.5 (Category of non-skeletal cyclic cooperads)**.** We denote by $Cy\mathcal{C}oop_{\mathcal{B}ij}$ the category of non-skeletal cyclic cooperads and cooperad maps that commute with the actions of bijections, cocomposition and counits.

4.8.2 Equivalence of definitions

We have functors $E^{\#}_{cyco}$ and $R^{\#}_{cyco}$ on the cooperad categories.

$$E^{\#}_{cyco} : Cy\mathcal{C}oop_{\Sigma} \rightleftarrows Cy\mathcal{C}oop_{\mathcal{B}ij} : R^{\#}_{cyco}$$

Theorem 4.8.2.1. *There is an equivalence of categories $Cy\mathcal{C}oop_{\mathcal{B}ij} \to Cy\mathcal{C}oop_{\Sigma}$.*

Proof. The proof is dual to the proof of Theorem 3.4.0.2 for cyclic operads. □

Remark 4.8.2.2. We may also obtain an equivalence of the skeletal cyclic and non-skeletal cyclic cooperad definitions by the fact a cyclic cooperad in a category $\mathcal{C}$ is a cyclic operad in $\mathcal{C}^{op}$.

We therefore have an extension of diagram (4.1) to include the equivalence for cyclic cooperads. In the diagram, all vertical arrows are forgetful functors and horizontal arrows are equivalences.

$$\begin{array}{ccc}
Cy\mathcal{C}oop_{\mathcal{B}ij} & \underset{R^{\#}_{cyco}}{\overset{R^{\#}_{cyco}}{\rightleftarrows}} & Cy\mathcal{C}oop_{\Sigma} \\
\downarrow V' & & \downarrow V' \\
\mathcal{C}oop_{\mathcal{B}ij_*} & \underset{R^{\#}_{co}}{\overset{E^{\#}_{co}}{\rightleftarrows}} & \mathcal{C}oop_{\Sigma_*} \\
\downarrow V & & \downarrow V \\
Fun(\mathcal{B}ij_*^{op}, \mathcal{C}) & \underset{R^{*}}{\overset{E^{*}}{\rightleftarrows}} & Fun(\Sigma_*^{op}, \mathcal{C})
\end{array}$$

4.8.3 Examples

- The coassociative cooperad is cyclic.
- The cocommutative cooperad is cyclic.

Chapter 5

Trees and partitions

In this chapter, we explore a number of examples of 'hidden' symmetric group actions on trees, partitions and finite sets. The motivation for this comes from the action of Σ_{n+1} on a space of trees studied by Robinson and Whitehouse in [RW96]. In this paper, they show that the representation of Σ_{n+1} corresponding to the natural action on their space of fully grown n-trees is related to the Lie representation.

In [Rob04], it is shown that these results in the tree space correspond to similar results for partitions due to a Σ_n-equivariant homeomorphism of spaces. This result leads to the question of what the Σ_{n+1} action looks like in the poset of partitions. The additional action is not obvious in this case, however there is a simpler action of Σ_{n+1} on a subset of partitions, giving a representation that can be decomposed into irreducibles. We explore this example of a hidden action and use representation theory to study its properties.

Following this, we explore a couple of similar hidden actions, on collections of finite ordered sets of fixed size.

5.1 Background

We will begin by covering some relevant background material that we will need for this chapter.

5.1.1 Partitions

In this chapter we will be referring to both partitions of finite sets, and partitions of integers. We will set out notation for both in this section in order to avoid confusion.

Recall from Definition 1.5.0.2 that we denote an integer partition of n by $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. We denote by $\underline{n}$ the set $\{1, 2, \dots, n\}$, for $n \in \mathbb{N}$.

Definition 5.1.1.1 (Partition of $\underline{n}$)**.** A partition of the set $\underline{n}$ is a collection of nonempty disjoint subsets U_i of $\underline{n}$ such that $\bigcup_i U_i = \underline{n}$.

Then the *shape* of such a partition is the collection of sizes of the subsets U_i, written in the form $(\#U_i, \#U_2, ...)$.

5.1.2 Simplicial sets

A particularly useful construction that allows one to pass between sets and topological spaces is geometric realisation. We will later see a topological space built from set partitions in this way.

Definition 5.1.2.1 (Simplicial set)**.** A simplicial set X is a sequence of sets $X_1, X_2, \ldots, X_n$ along with a collection of functions $d_i : X_n \mapsto X_{n-1}$ and $s_i : X_{n+1} \mapsto X_n$. We call the s_i face maps and the d_i degeneracy maps. These maps are subject to a number of conditions that can be found in [Fri12, Definition 3.2].

Definition 5.1.2.2 (Geometric realisation, [Fri12, Definition 4.1])**.** Let X be a simplicial set. Then the *geometric realisation* $|X|$ of X is defined by

$$|X| := \coprod_{n=0}^{\infty} X_n \times |\Delta^n| / \sim,$$

where the equivalence relation $\sim$ is generated by

$$\begin{aligned} (x, D_i(p)) &\sim (d_i(x), p) \quad \text{for } x \in X_{n+1} \text{ and } p \in |\Delta^n|, \\ (x, S_i(p)) &\sim (s_i(x), p) \quad \text{for } x \in X_{n-1} \text{ and } p \in |\Delta^n|, \end{aligned}$$

where D_i and S_i are face inclusions and collapses.

Note that $|X|$ is a CW complex and has one n-cell for every (non-degenerate) n-simplex of X.

Definition 5.1.2.3 (Nerve)**.** The *nerve* $N(\mathcal{C})$ of a category $\mathcal{C}$ is a simplicial set such that $N(\mathcal{C})_k$ is given by the k-tuples

$$f_0 \to f_1 \to \cdots \to f_k$$

of composable morphisms φ_i in $\mathcal{C}$. Then we define the face maps $d_i : N(\mathcal{C})_k \to N(\mathcal{C})_{k-1}$ to be the map that takes

$$f_0 \to \cdots \to f_k$$

to the $(k-1)$-tuple omitting the i^{th} morphism:

$$f_0 \to \cdots \to \hat{f_i} \to \cdots \to f_k.$$

Similarly, the degeneracy maps $s_i : N(\mathcal{C})_k \to N(\mathcal{C})_{n+1}$ inserts the identity position in the *ith* position of a k-tuple to get a $(k+1)$-tuple.

Then we may take the geometric realisation of a nerve $N(\mathcal{C})$, giving a topological space.

5.2 The space of fully grown n-trees, T_n

Robinson and Whitehouse [RW96] introduce the space of fully grown n-trees and the associated Σ_{n+1} representation on its reduced homology. We summarise some of these results, in which some terminology is from [Boa71].

A tree is a contractible graph. We will refer to both the vertices with degree one and their connecting edges as *leaves*, and all other vertices are sometimes referred to as nodes, or internal vertices. An edge connecting two nodes is called an *internal edge*. Define an n-tree to be a tree with $n+1$ leaves labelled $0, 1, \ldots, n$, where the leaf labelled 0 is called the *root*. We parametrise these trees with edge lengths $0 < l(\alpha) \leq 1$ for internal edge α, and leaves have fixed length 1.

Let $\widetilde{T}_n$ denote the space of isomorphism classes of n-trees, which is a cubical complex. Define a tree to be *fully grown* if it has at least one internal edge of maximal length 1. We consider the space T_n of fully grown n-trees. Then T_n is a subspace of $\widetilde{T}_n$, and in fact $\widetilde{T}_n$ is a cone with base T_n. This can be seen by shrinking continuously and linearly all the internal edges in a fully grown tree. The vertices of T_n are the trees with just one internal edge of length 1.

One can triangulate T_n as a simplicial complex of pure dimension $n-3$, where each $(n-4)$-simplex is a face of exactly three $(n-3)$-simplices.

Remark 5.2.0.1. An alternative formulation of the space T_n is one where the sum of lengths of internal edges is 1, instead of having at least one internal edge of length 1. In this case, there can only be at most one internal edge of length 1. This gives T_n the structure of a simplicial complex, rather than a cubical complex.

Proposition 5.2.0.2 ([RW96, Theorem 1.5])**.** *The space of fully grown n-trees is homotopy equivalent to a wedge of spheres. That is,*

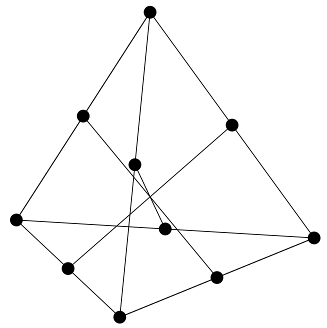
$$T_n \simeq \bigvee_{(n-1)!} S^{n-3}.$$

Therefore, the reduced (co)homology is concentrated in dimension $n-3$.

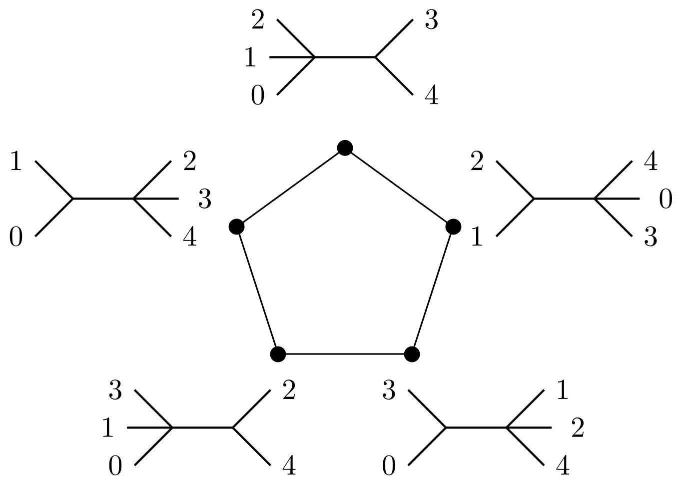
Example 5.2.0.3 ($n = 4$)**.** We have that

$$T_4 \simeq \bigvee_{6} S^1.$$

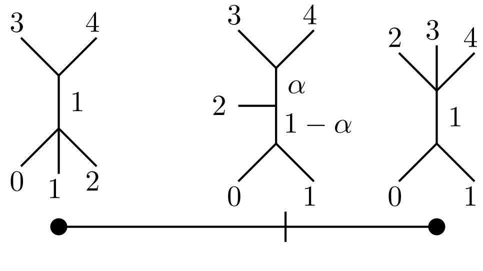
The space is the wedge of six circles, and can in fact be depicted by the diagram below. That is, the one-skeleton of a tetrahedron, with opposite midpoints of edges connected by three further edges. This has ten vertices as expected, as there are ten fully grown 4-trees.



Then we can view one copy of S^1 in this picture as a loop with five vertices such as the below diagram.



The 0-simplies are given by trees with one internal edge of length 1, and can be viewed as having had all other internal edge lengths collapsed to 0. The points along each edge represent trees where an internal edge is lengthened to pass between the two vertex trees. For example, see the edge below. An edge in T_n has length 1, and at a point of distance α along the edge, we have a tree with one internal edge grown from length 0 to length α, and the other with distance that has shrunk from length 1 to length $1-\alpha$. In this way, the leaf labelled 2 'travels' from one end to the other of the vertex trees.



5.3 Symmetric group action in T_n

The symmetric group Σ_{n+1} acts on T_n by permuting the labels of all the leaves as well as the root, whereas the symmetric groups Σ_n and Σ_{n-1} permute the labels $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, n-1\}$ respectively. Therefore, it is clear that the Σ_{n+1} action restricts to the actions of the smaller symmetric groups by fixing the extra one or two leaf labels.

Proposition 5.3.0.1 ([RW96, Theorem 3.1])**.** *The representation of Σ_{n+1} on the reduced homology group $\tilde{H}_{n-3}(T_n)$ has character given by*

$$sgn \cdot (Ind_{\Sigma_n}^{\Sigma_{n+1}} Lie_n - Lie_{n+1}),$$

and is a dimension $(n-1)!$ representation. Robinson and Whitehouse call this the tree representation. Here, Lie_n gives the character of the Lie representation defined in [RW96], and sgn is the alternating character.

This representation is determined by considering fixed point sets of the Σ_{n+1} action, and calculating the Euler characteristics of these sets. Then the characters of the representation can be determined from the Euler characteristics, and these coincide with the characters of $sgn \cdot (Ind_{\Sigma_n}^{\Sigma_{n+1}} Lie_n - Lie_{n+1})$.

The restriction of the tree representation to Σ_{n-1} is isomorphic to the regular representation, and the restriction of the tree representation to Σ_n has character $sgn \cdot Lie_n$.

5.4 The poset of non-trivial partitions Λ_n

In [Rob04], Robinson proves a relationship between the tree space and the poset of non-trivial partitions of $\underline{n} := \{1, 2, \ldots, n\}$, and results regarding symmetric group representations on the (co)homology of this space. We explore these results in order to understand the hidden action of Σ_{n+1} in the setting of partitions of $\underline{n}$.

Partitions of $\underline{n}$ form a poset ordered by refinement. The whole set $\underline{n}$ and the partition $\{1\}\{2\}\ldots\{n\}$ of singletons are defined as trivial partitions. Then, we will consider the poset Λ_n of non-trivial partitions of the set $\underline{n}$. This poset has the ordering that for partitions p and p', $p \leq p'$ if p is finer than p'. That is, if each block in p is a subset of a block in p'.

Example 5.4.0.1 ($n = 4$)**.**

- $\{1, 2\}\{3\}\{4\} < \{1, 2, 3\}\{4\}$
- $\{1, 3\}\{2, 4\} < \{1, 2, 3, 4\}$, however $\{1, 2, 3, 4\}$ is a trivial partition and therefore does not belong to Λ_4
- Neither of $\{1, 2, 3\}\{4\}$ and $\{1\}\{2, 3, 4\}$ is finer.

Note that if we were to include the trivial partitions, we would then have the structure of a complete lattice [BS81, Theorem 4.11], as every subset now has its greatest lower bound and least upper bound included in the poset.

5.5 Relationship between T_n and Λ_n

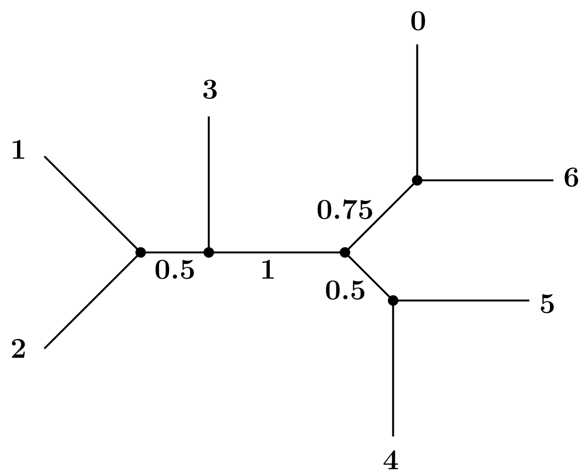
There is a Σ_n-equivariant homeomorphism between the space T_n and the nerve of the poset Λ_n, described in more detail in [Rob04, Proposition 2.7]. Here, as in [Rob04], we use the notation Λ_n to mean both the poset of partitions and the nerve of this poset, which is a simplicial complex.

Roughly speaking, given a tree in T_n, the homeomorphism is as follows. We define a family of maps $\{\gamma_i(t)|1 \leq i \leq n\}$ to be the paths in the tree starting at the root and ending at the vertex labelled i. The paths are unique since trees do not contain any loops. When t is sufficiently small, all $\gamma_i(t)$ are equal, and for t sufficiently large all $\gamma_i(t)$ will be different, since the paths diverge. Then for any $t \in [0, 1]$ we define the partition p_t to be the partition of $\underline{n}$ with

$$i \sim j \iff \gamma_i(t) = \gamma_j(t),$$

and $i \sim j$ if and only if i and j are contained in the same block of the partition. This can be extended to the whole of Λ_n by considering a particular parametrisation of $\gamma_i(t)$ and using barycentric coordinates.

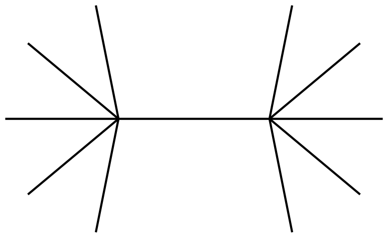
Example 5.5.0.1. The tree pictured below is a fully grown 6-tree. Recall that leaves are always given length 1. Then this tree corresponds to the sequence of non-trivial partitions $\{1, 2, 3, 4, 5\}\{6\} > \{1, 2, 3\}\{4, 5\}\{6\} > \{1, 2, 3\}\{4\}\{5\}\{6\} > \{1, 2\}\{3\}\{4\}\{5\}\{6\}$.



5.6 Σ_n action on partitions

There is an obvious Σ_n action on the set of partitions of $\underline{n}$ by simply permuting elements.

Since there is a homeomorphism between the tree space T_n and Λ_n, the Σ_{n+1} action on T_n described in [RW96] carries over to the partition space. In particular, this action restricts to an action on the set of vertices of the tree complex, but it is not an action directly on the set of partitions of $\underline{n}$, only on the nerve of the poset as a whole. This is because some partitions in Λ_n correspond to barycentres of simplices in the triangulation of the tree space, rather than vertices. These partitions are mapped to other midpoints by the Σ_{n+1} action, and thus not to a single partition, but to a sequence of partitions. Therefore, we will instead consider the symmetric group action on an appropriate subset of partitions of $\underline{n}$ that correspond with the vertices of the space T_n. That is, trees with one internal edge as shown by the below diagram.



5.6.1 Σ_n action in Λ_n

We will introduce the following notation for subsets of fixed size, where $n \geq 1$ and $1 \leq m \leq n$,

$$S_m^n := \{A \subset \underline{n} : |A| = m\}.$$

We will also use the following notation for partitions of $\underline{n}$ of a particular shape,

$$P_m^n := \{\textit{partitions with shape } (m, 1^{n-m})\}.$$

Then, there is a bijection σ

$$\begin{aligned} S_m^n &\longleftrightarrow P_m^n \\ \sigma : A &\longmapsto A\{b_1\} \dots \{b_{n-m}\}, \end{aligned}$$

where $\{b_1, \dots, b_{n-m}\} = \overline{A} := \underline{n} \setminus A$.

Furthermore, we also have a bijection between S_m^n and S_{n-m}^n:

$$\begin{gathered} A \longleftrightarrow \overline{A} \\ \tau : A\{b_1\} \dots \{b_{n-m}\} \mapsto \{b_1, b_2, \dots, b_{n-m}\}\{a_1\} \dots \{a_m\}, \end{gathered}$$

where $\{a_1, \dots, a_m\} = A$.

Proposition 5.6.1.1. *The bijections σ and τ defined above are Σ_n-equivariant.*

Proof. The first of these bijections is clearly Σ_n-equivariant since for $\pi \in \Sigma_n$,
$\pi\overline{A} = \overline{\pi A}$, and therefore

$$\pi(\sigma A) = \{\pi A\}\{\pi(b_1)\} \dots \{\pi(b_{n-m})\} = \{\pi A\}\{\tilde{b}_1\} \dots \{\tilde{b}_{n-m}\} = \sigma(\pi A),$$

where $\tilde{b}_i \in \overline{\pi A}$.

The second is Σ_n-equivariant because for $\pi \in \Sigma_n$,

$$\begin{aligned} \pi(\tau(A\{b_1\} \dots \{b_{n-m}\})) &= \pi(\{b_1, \dots, b_{n-m}\}\{a_1\} \dots \{a_m\}) \\ &= \{\pi(b_1), \dots, \pi(b_{n-m})\}\{\pi(a_1)\} \dots \{\pi(a_m)\} \\ &= \tau(\pi(A\{b_1\} \dots \{b_{n-m}\})). \end{aligned}$$

□

Definition 5.6.1.2. We define the notation

$$S^n := \bigcup_{2 \le m \le n-1} S^n_m,$$
$$P^n := \bigcup_{2 \le m \le n-1} P^n_m.$$

Then P^n is the set of partitions that correspond to the collection of sets S^n. These are precisely the partitions that correspond to the vertices in the tree space T_n.

5.6.2 The Σ_n representation

We will now give the representation of Σ_n given by the permutation action on P^n_m, both as an induced representation and explicitly in terms of Specht modules. The key components of the below proposition and its proof, in particular the decomposition into irreducibles, are discussed by Wildon in [Wil14]. The representation theory notation used is included in Chapter 1.

Proposition 5.6.2.1 (The Σ_n representation)**.** *The representation of Σ_n on P^n_m is the induced representation of the trivial representation on $\Sigma_m \times \Sigma_{n-m}$. That is,*

$$\mathbb{C}P^n_m \cong 1 \uparrow^{\Sigma_n}_{\Sigma_m \times \Sigma_{n-m}} \cong M^{(m,n-m)},$$

with decomposition into irreducible representations

$$S^{(m,n-m)} \oplus S^{(m+1,n-m-1)} \oplus \cdots \oplus S^{(n-1,1)} \oplus S^{(n)}.$$

Outline of proof. It is a result [Sag01] that if there is a non-zero homomorphism $\theta : S^\lambda \to M^\mu$ of $\mathbb{C}\Sigma_n$-modules that extends to a homomorphism $\tilde{\theta} : M^\lambda \to M^\mu$, then $\lambda \trianglerighteq \mu$.

This implies that the permutation module M^μ is a direct sum of Specht modules for partitions λ with $\lambda \trianglerighteq \mu$. Thus,

$$M^{(m,n-m)} \cong S^{(m,n-m)} \oplus S^{(m+1,n-m-1)} \oplus \cdots \oplus S^{(n-1,1)} \oplus S^{(n)}$$

for any m with $2m \le n$.

Furthermore, if $\lambda = (m, n-m)$, by ignoring the second row of each $(m, n-m)$-tabloid, we get a bijection from the m-subsets of $\underline{n}$ and λ-tabloids. Thus, $M^{(m,n-m)}$ is isomorphic to the $\mathbb{C}\Sigma_n$-permutation module of the action of Σ_n on subsets of $\underline{n}$ of size m. This is exactly the representation we are interested in, for the action of Σ_n on the partitions of shapes $(m, 1^{n-m})$, as was shown by our bijection to the sets S^n_m. □

The dimension of the above Σ_n representation is $\binom{n}{m}$. This can be calculated using the formula for the dimension of a permutation module given in Definition 1.5.1.3, or by the fact that the dimension of a representation associated with a group action on a vector space is simply the dimension of the vector space (see Definition 1.4.0.1).

$$\dim M^{(m,n-m)} = \frac{n!}{m!(n-m)!} = \binom{n}{m} = \#P^n_m.$$

Example 5.6.2.2 ($n = 5,\ m = 2$)**.** In this case the dimension of the representation is $\binom{5}{2} = 10$. The decomposition into irreducible representations is

$$S^{(3,2)} \oplus S^{(4,1)} \oplus S^{(5)}.$$

5.7 Σ_{n+1} action on partitions

The action of Σ_{n+1} on P^n can be described as follows. The elements of P^n are of the form

$$\{a_1, a_2, \dots, a_m\}\{b_1\}\{b_2\}\dots\{b_{n-m}\},$$

with shape $(m, 1^{n-m})$ for any $2 \leq m \leq n-1$.

We will explicitly describe a Σ_{n+1} action on S^n which transfers to P^n via the bijection

$$P^n \leftrightarrow \bigcup_{2 \leq m \leq n-1} S^n_m = S^n.$$

This restricts to the natural Σ_n action from the previous section, and we will check directly that it is well-defined.

Definition 5.7.0.1. We extend the natural Σ_n action to an action of the symmetric group Σ_{n+1} on S^n as follows. Let $\pi \in \Sigma_{n+1}$ be any permutation of elements of $\{0, 1, \dots, n\}$ and $A \subset \underline{n}$ a set in S^n. Then

$$\pi : A \mapsto \begin{cases} \pi A & \text{if } \pi \in \Sigma_{n+1} \setminus \Sigma_n,\ \pi^{-1}(0) \notin A, \\ \pi \overline{A} \cup \{\pi(0)\} & \text{if } \pi \in \Sigma_{n+1} \setminus \Sigma_n,\ \pi^{-1}(0) \in A, \end{cases}$$

where $\overline{A} = \underline{n} \setminus A$.

Proposition 5.7.0.2. *The Σ_{n+1} action in definition 5.7.0.1 is well-defined.*

Proof. This is a case by case check, which can be found in Appendix A. □

Example 5.7.0.3 ($n = 5, m = 3$)**.** Let $A = \{1, 3, 4\}$. Then

$$(3,5)A = \{1,5,4\}$$
$$(0,1,2)A = \{2,3,4\}$$
$$(1,2)(3,0)A = \{1,5,3\}.$$

Remark 5.7.0.4. Note there is a pairing between the shapes associated with S^n_m and S^n_{n-m+1}. That is, elements of S^n_m are mapped to either elements of S^n_m or of S^n_{n-m+1} by the Σ_{n+1} action, and the same for elements of S^n_{n-m+1}. Therefore, the sets $S^n_m \cup S^n_{n-m+1}$ are Σ_{n+1}-invariant.

5.7.1 The Σ_{n+1} representation

Proposition 5.7.1.1 (Representation dimension)**.** *The dimension of the representation of* Σ_{n+1} *on* P^n *is*

$$\#P^n = 2^n - (n+2).$$

Proof. There are $n-2$ possible shapes of partition of the relevant kind, corresponding to the number m of singletons in the partition. For each of these there are $\binom{n}{m}$ partitions, and therefore we have

$$\begin{aligned}\#P^n &= \sum_{m=2}^{n-1} \binom{n}{m} \\ &= 2^n - \binom{n}{n} - \binom{n}{1} - \binom{n}{0} \\ &= 2^n - (n+2).\end{aligned}$$

□

In the proposition below, we give the unique Σ_{n+1} representation that restricts to the Σ_n representation on P^n_m which is that associated to the restriction of the Σ_n action on P^n to P^n_m.

Remark 5.7.1.2. We have seen that the action of Σ_{n+1} is on pairs P^n_m and P^n_{n-m+1}. However, the irreducible decomposition of the Σ_{n+1} representation does indeed split according to value of m. We see this by noticing that the Σ_n representations for fixed m are each the restriction of a single unique Σ_{n+1} representation.

Proposition 5.7.1.3 (The Σ_{n+1} representation)**.** *For the* P^n_m *with* $2m \geq n$*, but not for odd* $n \geq 5$ *with* $m = \frac{n+1}{2}$ *or* $m = \frac{n-1}{2}$*, the* Σ_n *representation on* P^n_m *is the restriction of a representation of* Σ_{n+1}*, uniquely and with decomposition*

$$\begin{aligned}&\bigoplus_{k=0}^{k \leq \frac{n-m}{2}} S^{(m+1+2k, n-m-2k)} \\ &= S^{(m+1,n-m)} \oplus S^{(m+3,n-m-2)} \oplus \dots\end{aligned}$$

where S^λ *denotes the Specht module associated with shape* λ*.*

Outline of proof. Recall from Proposition 5.6.2.1 that P^n_m has the Σ_n representation

$$M^{(m,n-m)} \cong S^{(m,n-m)} \oplus S^{(m+1,n-m-1)} \oplus \dots \oplus S^{(n-1,1)} \oplus S^{(n)}.$$

If we assume that $2m \geq n$ then we have only one choice for the Σ_{n+1} representation this is a restriction of. It is clear that any potential modules must be associated to tableaux of no more than two rows, otherwise their restriction will include those with more than two rows which do not appear here.

Taking $S^{(m,n-m)}$ we see that possible modules containing this in their restriction are those associated to the partitions $(m+1, n-m)$ or $(m, n-m+1)$. The latter would give an additional summand on the restriction of $S^{(m-1,n-m+1)}$ which we do not want, and so it must be the first. This also gives us the second term in the sum, so we next consider the third, namely $S^{(m+2,n-m-2)}$. This process continues until we reach one of $S^{(n+1)}$ or $S^{(n,1)}$ as the final module in the decomposition. Therefore, everything is uniquely determined. □

Proposition 5.7.1.4 (The outstanding cases with non-unique extension to Σ_{n+1})**.** *If n is of the form $4l+1$ for some $l \in \mathbb{N}$, and $m = \frac{n+1}{2}$, then the Σ_{n+1} representation on P^n_m is given by*

$$\bigoplus_{k=0}^{k \leq \frac{n-m}{2}} S^{(m+1+2k,n-m-2k)}. \tag{5.1}$$

If n is of the form $4l+1$ and $m = \frac{n-1}{2}$, or if n is of the form $4l+3$ for some $l \in \mathbb{N}$ and $m = \frac{n+1}{2}$ or $m = \frac{n-1}{2}$, then the representation is of the form

$$\bigoplus_{k=0}^{k < \frac{n-m}{2}} S^{(m+2+2k,n-m-(2k+1))}. \tag{5.2}$$

Note that in the cases $n = 4l+1$, the representations for $m = \frac{n+1}{2}$ and $m = \frac{n-1}{2}$ are the same.

Proof. First, note that by considering which Young tableaux restrict to the required tableaux for the Σ_n representation, (5.1) and (5.2) are the only possible Σ_{n+1} representations.

We may choose one suitable (one with a different character under (5.1) and (5.2)) generating permutation to calculate the characters in order to determine which of (5.1) or (5.2) is correct.

Such a permutation is the $(n+1)$-cycle $(1, 2, \dots, n, n+1)$. By using the Murnaghan-Nakayama rule, we see that in the decompositions of (5.1) and (5.2), the only terms that will contribute something non-zero to the character for an $(n+1)$-cycle are the last terms. These correspond to Young tableaux with either one or two rows, and a tableau with one row gives a character of $+1$, whereas a tableau of two rows gives a character of -1. In (5.1), the character is $+1$ for odd m and is -1 for even m. In (5.2) the character is $+1$ for even m and is -1 for odd m. Since the symmetric group action is isomorphic to an action that permutes the vertices of trees, we need a representation with no negative characters.

We deal with the possible values for n and m separately. We first consider the cases where $m = \frac{n+1}{2}$ since for any n this shape of partition pairs with itself because $n-m+1 = m$ and so the Σ_{n+1} action is internal. When $n = 4l+1$ we have that $m = \frac{n+1}{2}$ is odd, so that the character for (5.1) is $+1$ and therefore this is the correct representation. Similarly, when $n = 4l+3$, we have that $m = \frac{n+1}{2}$ is even, so that the character for (5.2) is $+1$ and so (5.2) is the correct representation.

Next we consider the case $n = 4l+1$ and $m = \frac{n-1}{2}$. The symmetric group Σ_{n+1} acts on the pair P^n_m and P^n_{n-m+1}. For $n = 4l+1$, $m = \frac{n-1}{2}$ is even, and so is $n-m+1$. Furthermore, we know from earlier that P^n_{n-m+1} has the unique representation of (5.1) and so gives a character of -1. Therefore, to get a non-negative character overall we need character $+1$ for P^n_m and so the corresponding representation must be (5.2).

Finally, we have $n = 4l + 3$, $m = \frac{n-1}{2}$. Here, m is odd, and hence $n - m + 1$ is odd. The representation of P^n_{n-m+1} therefore has character $+1$ and so the possible overall characters are 0 or $+2$. But we know from the lack of symmetry in the trees associated with these shapes of partition that none are fixed by an $(n+1)$-cycle and so the character for P^n_m must be -1. Thus, the correct representation is (5.2). □

Example 5.7.1.5. Here are some decompositions for different values of m and n.

- Case 1: $n = 6, m = 3$. The Σ_n representation decomposes as $S^{(4,3)} \oplus S^{(6,1)}$.
- Case 2: $n = 5, m = 3$. The Σ_n representation decomposes as $S^{(4,2)} \oplus S^{(6)}$.
- Case 3: $n = 5, m = 2$. The Σ_n representation decomposes as $S^{(4,2)} \oplus S^{(6)}$.

Remark 5.7.1.6. Note that the Σ_{n+1} representation for P^n_m with $2m \geq n$ is the same as for P^n_{n-m} due to the symmetry in the Σ_n representation and uniqueness of associated induced Σ_{n+1} representation.

5.8 Symmetric group action on ordered subsets

As we have previously seen, the action of Σ_{n+1} on certain partitions is equivalent to an action on subsets of $\underline{n}$. Also, we see from the decomposition into irreducible representations that the Σ_{n+1} representation does indeed split according to each value of m. Therefore, we are led to believe that there may exist an action on m-sets of fixed size which gives the same Σ_n and Σ_{n+1} representations as was given by the action on trees and partitions. However, when trying to construct such an action, we instead found a Σ_n action on fixed sized ordered sets with a hidden Σ_{n+1} action that gives a representation of the symmetric group different to the one we have seen previously.

5.8.1 Σ_n action

In this section, we will work with $\mathbb{Z}$-linear combinations of equivalence classes of ordered sets. This is because the symmetric group action introduces signed ordered sets and sums of these.

Define by

$$O^n_m := \{\text{ordered } A \subset \underline{n} : |A| = m\},$$

the collection of ordered subsets of $\underline{n}$ with fixed size m, and extend linearly to the module $\mathbb{Z}[O^n_m]$.

We say an ordered set of natural numbers A is in its *natural order* if $A = (a_1, a_2, \dots, a_m)$, where $a_1 < a_2 < \dots < a_m$. Then for any given ordered set $A = (a_1, \dots, a_m)$ not necessarily in the natural order, there is a unique permutation $\pi_A \in \Sigma_A$ where $\Sigma_A = \{\textit{bijections } A \to A\}$, such that $\pi_A A$ is in the natural order. That is, $\pi_A a_1 < \pi_A a_2 < \dots < \pi_A a_m$.

Note that $\Sigma_A \subset \Sigma_n$, and we may view permutations in Σ_A as permutations in Σ_n by extending with the identity on the complement $\overline{A}$ in $\underline{n}$.

Definition 5.8.1.1. We define an equivalence relation on $\mathbb{Z}[O^n_m]$ by

$$A \sim (sgn\pi_A)\pi_A A$$

and extending to linear combinations. We write $[A]$ for the equivalence class of $A \in O^n_m$.

Proposition 5.8.1.2. *The above equivalence relation is well-defined.*

Proof. If we equivalently formulate the equivalence relation as

$$A \sim B \text{ if and only if } (sgn\pi_A)\pi_A A = (sgn\pi_B)\pi_B B,$$

then we can see that it is reflexive, symmetric and transitive. □

The symmetric group Σ_n acts on O^n_m by permutation of elements. That is, for any ordered set $A \in O^n_m$,

$$\sigma \cdot A = \sigma\{a_1, a_2, \ldots, a_m\} = \{\sigma a_1, \sigma a_2, \ldots, \sigma a_m\},$$

and this extends linearly to $\mathbb{Z}[O^n_m]$.

Proposition 5.8.1.3. *The symmetric group Σ_n action on $\mathbb{Z}[O^n_m]/\sim$ given by permutation of set elements and extending $\mathbb{Z}$-linearly is a well-defined action.*

Proof. We have the obvious action Σ_n on $\mathbb{Z}[O^n_m]$ by permutation of set elements mentioned above. It remains to check that this extends to $\mathbb{Z}[O^n_m]/\sim$ by checking that the action is well-defined on equivalence classes. In particular, we need that for all permutations $\tau \in \Sigma_n$,

$$\tau A \sim \tau(sgn\pi_A)\pi_A A, \tag{5.3}$$

or equivalently

$$\begin{array}{c}[(\tau a_1, \tau a_2, \ldots, \tau a_m)] \\ \shortparallel \\ [(sgn\pi_{\tau A})\pi_{\tau A}\tau A]\end{array} = [(sgn\pi_A)\tau\pi_A A].$$

On the left-hand side, we have that the elements $\tau a_i \in \pi_{\tau A}\tau A$ are in natural order. To get elements in their natural order in the equivalence class on the right, let us rewrite the right-hand side as

$$[(sgn\pi_A)\tau\pi_A A] = [(sgn\pi_A)(sgn\pi_{\tau\pi_A A})\pi_{\tau\pi_A A}\tau\pi_A A].$$

Now note that $\pi_{\tau\pi_A A}\tau\pi_A A = \pi_{\tau A}\tau A$. Therefore, it remains to check that

$$sgn\pi_{\tau A} = (sgn\pi_A)(sgn\pi_{\tau\pi_A A}).$$

The permutation $\pi_{\tau A}$ sorts the ordered set τA into its natural order, and similarly $\pi_{\tau\pi_A A}$ is the permutation sorting the ordered set $\tau\pi_A A$ into natural order.

We have the following diagram.

$$
\begin{array}{ccccc}
A & \xrightarrow{\tau} & \tau A & \xrightarrow{\pi_{\tau A}} & \pi_{\tau A}\tau A \\
{\scriptstyle \pi_A}\downarrow & & & & \| \\
\pi_A A & \xrightarrow[\tau]{} & \tau\pi_A A & \xrightarrow[\pi_{\tau\pi_A}]{} & \pi_{\tau\pi_A A}\tau\pi_A A
\end{array}
$$

This diagram commutes, since applying τ and then sorting gives the same ordered set as sorting first and then applying τ and sorting again. Then we have that

$$\pi_{\tau A} = \pi_{\tau\pi_A A}\tau\pi_A\tau^{-1},$$

and therefore we get

$$
\begin{aligned}
sgn\pi_{\tau A} &= (sgn\pi_{\tau\pi_A A})(sgn\tau)(sgn\pi_A)(sgn\tau^{-1}) \\
&= (sgn\pi_{\tau\pi_A A})(sgn\pi_A)
\end{aligned}
$$

as required. □

5.8.2 Σ_{n+1} action

To show that we have a Σ_{n+1} action on these sets, we can show that we have an extension of the Σ_n action. Using the fact that the symmetric group can be generated by adjacent transpositions t_i, we only need to consider the extra transposition t_n and check that the relevant relations hold.

We denote by A_a^n the ordered set where the element $a \in A$ is replaced by n for $n \notin A$.

Definition 5.8.2.1. The additional transposition that extends the Σ_n action on $\mathbb{Z}[O_m^n]$ to an action of Σ_{n+1} acts as follows,

$$t_n : [A] \mapsto \begin{cases} -[A] & \text{if } n \in A, \\ [A] - \sum\limits_{a \in A} [A_a^n] & \text{if } n \notin A. \end{cases}$$

Example 5.8.2.2 ($n = 4, m = |A| = 3$)**.** We have the action of the transposition $t_4 = (4, 5)$:

$$
\begin{aligned}
t_4[(1,2,3)] &= [(1,2,3)] - [(4,2,3)] - [(1,4,3)] - [(1,2,4)] \\
&= [(1,2,3)] - [(2,3,4)] + [(1,3,4)] - [(1,2,4)].
\end{aligned}
$$

Proposition 5.8.2.3. *The above is a well-defined action of* Σ_{n+1} *on* $\mathbb{Z}[O_m^n]/\sim$.

Proof. This is a case by case analysis and can be found in Appendix A □

Proposition 5.8.2.4 (Representation dimension)**.**

$$\dim \mathbb{Z}[O_m^n]/\sim = \binom{n}{m} = \#P_m^n$$

Note that the representation $\mathbb{Z}[O_m^n]/\sim$ of Σ_{n+1} is not isomorphic to the Σ_{n+1} representation P_m^n.

5.8.3 The Σ_n and Σ_{n+1} representations

We show by determining the representation given by the action on $\mathbb{Z}[O^n_m]/\sim$ that it is indeed a different hidden action to the one on P^n.

Example 5.8.3.1 (The case $n = 4$)**.** The non-trivial cases to consider are $m = 3$, $m = 2$. We determine the representation by considering the matrices of generating adjacent transpositions acting on the relevant sets of size m.

For $m = 3$, there is a $\mathbb{Z}$-basis

$$\{[(a_1, a_2, a_3)], [(a_1, a_2, a_4)], [(a_1, a_3, a_4)], [(a_2, a_3, a_4)]\}$$

of equivalence classes of ordered sets.

The generating matrices are as follows.

$$(12) : \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (23) : \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$(34) : \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (45) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

Recall from Definition 1.4.1.2 that the character χ is given by the trace of the relevant matrix. Then these all give a character of -2.

By multiplying the relevant matrices, we get the characters for all cycle types of permutations:

e	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
4	-2	0	1	1	0	-1

Hence, by comparison with the Σ_4 character table, the Σ_4 representation $\mathbb{Z}[O^4_3]/\sim$ is $S^{(2,1,1)} \oplus sgn$, and the Σ_5 representation is $S^{(2,1,1,1)}$.

By the same process, for $m = 2$ we get the characters

e	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
6	0	-2	0	0	0	1

Therefore, the Σ_4 representation $\mathbb{Z}[O^4_2]/\sim$ is given by $S^{(3,1)} \oplus S^{(2,1,1)}$ and the Σ_5 representation is $S^{(3,1,1)}$.

This example and a few similar calculations for different n, gave an indication of the general form of the representations associated to the action.

Proposition 5.8.3.2. *The decompositions of the representations given by the* Σ_n *and extended* Σ_{n+1} *action on* $\mathbb{Z}[O^n_m]/\sim$ *into Specht modules are*

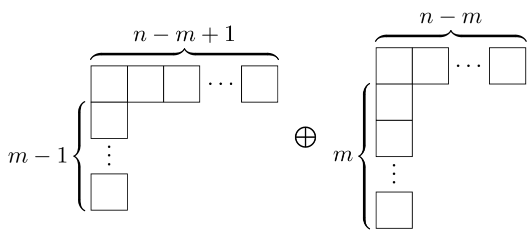
$$\Sigma_n : S^{(n-m+1,1,1,\ldots,1)} \oplus S^{(n-m,1,1,\ldots,1)},$$
$$\Sigma_{n+1} : S^{(n-m+1,1,1,\ldots,1)}.$$

The Σ_n and Σ_{n+1} representations above have dimension

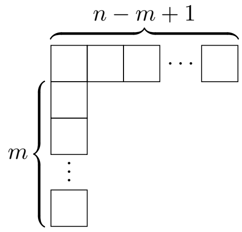
$$\dim\left(S^{(n-m+1,1,1,\ldots,1)} \oplus S^{(n-m,1,1,\ldots,1)}\right) = \binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$$
$$\dim\left(S^{(n-m+1,1,1,\ldots,1)}\right) = \binom{n}{m}.$$

Proof. The Young diagrams for the decompositions are below.

Σ_n :



Σ_{n+1} :



First note that the Σ_{n+1} representation is the unique representation that restricts to the Σ_n representation. Therefore, it is enough to prove that one of the above decompositions gives the correct representation.

If we focus on the Σ_{n+1} diagram, then the basis of standard tableaux of this shape are those with entries in $\{1, 2, \ldots, n+1\}$ which are increasing across rows and down columns. In particular, this is in bijection with the set of size m subsets of $\{2, 3, \ldots, n+1\}$. This is because the top left entry is required to be 1, meaning that the remaining entries are determined by the m-sized subset of $\{2, \ldots, n+1\}$ that fills the first column.

We re-index such that $i \mapsto i-1$ for all $i \in \{1, 2, \ldots, n+1\}$, that is, we now consider entries $\{0, 1, \ldots, n\}$. Then, if we denote by T a standard tableau of the above form, we have the bijection θ between such T and S^n_m.

$$T \overset{\theta}{\mapsto} \{\text{entries in column 1}\} \setminus \{0\} \text{ (assigned its natural order).}$$

Then we define the $\mathbb{Z}$-linear bijection

$$\tilde{\theta} : \mathbb{Z}\begin{bmatrix}\text{reindexed std tableaux of} \\ \text{shape } (n-m+1, 1^m)\end{bmatrix} \to \mathbb{Z}[O^n_m]/\sim$$

by

$$T \mapsto [\theta(T)]$$

and extending $\mathbb{Z}$-linearly.

We claim that this is Σ_n-equivariant.

We have that for a partition λ of $\underline{n}$, the Specht module S^λ has basis

$$\{\mathbf{e}_t : \text{t is a standard } \lambda\text{-tableau}\},$$

as in [Sag01, Theorem 2.5.2]. Here, $\mathbf{e}_t$ is a polytabloid. Then, it is also a result [Sag01, Chapter 2] that for $\sigma \in C_t$, where C_t is the column stabiliser of t, and for $\tau \in R_t$ with R_t the row stabiliser of t,

$$\sigma \mathbf{e}_t = (sgn\ \sigma)\mathbf{e}_t, \tag{5.4}$$

$$\tau \mathbf{e}_t = \mathbf{e}_t. \tag{5.5}$$

That is, elements of the column stabiliser act with sign, and elements of the row stabiliser act trivially on polytabloids.

For $\sigma \in \Sigma_n$ we require the diagram below to commute.

$$\begin{array}{ccc} \mathbf{e}_t & \xrightarrow{\quad\theta\quad} & [A] \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \sigma} \\ \sigma\mathbf{e}_t = \mathbf{e}_{\sigma t} & \xrightarrow[\theta]{} & sgn(\pi_{\sigma A})[\sigma A]. \end{array}$$

The right column is by definition of the Σ_n action in Proposition 5.8.1.3. The left column is a standard result, for example see [Sag01, Lemma 2.3.3]. The top map is simply applying the bijection $\tilde{\theta}$. Then it remains to check the bottom map, that is, that applying $\tilde{\theta}$ to $\mathbf{e}_{\sigma t}$ does indeed give $sgn(\pi_{\sigma A})[\sigma A]$.

The bijection $\tilde{\theta}$ maps standard tableaux to finite sets, and the tableau σt is not necessarily standard. We may order the elements of the first row of σt with a permutation $\tau \in R_1$. Similarly, we may order the elements of the first column of σt with a permutation $\pi \in C_1$. We note that the elements of the first column of σt in order are given by the set σA. Therefore, the permutation $\pi \in C_1$ that orders the elements of the first column, is precisely the permutation $\pi_{\sigma A}$. Then, by (5.5), we have that

$$\tau \mathbf{e}_{\sigma t} = \mathbf{e}_{\sigma t}$$

and by 5.4, we have

$$\pi_{\sigma A}\mathbf{e}_{\sigma t} = sgn(\pi_{\sigma A})\mathbf{e}_{\sigma t}.$$

Therefore, the diagram commutes as required.

We have a Σ_n action on the basis of tableaux of the Σ_{n+1} representation, isomorphic to the action on $\mathbb{Z}[O^n_m]$. This shows that the given decomposition for the Σ_{n+1} representation does indeed restrict to the correct Σ_n action. □

5.8.4 Extension to Λ_n

Finally, we explore an interesting question that arises from the action on sets. That is, the possibility of extending this action to an action analogous to the symmetric group action on the whole poset Λ_n of non-trivial partitions, not just the subset of partitions we have considered.

This is not as straightforward due to the different shapes of partitions. Before, we only considered the shapes that were in bijection with sets, allowing us to work with actions on sets in the natural way. When we extend to all partitions of $\underline{n}$, we have partitions with multiple nontrivial parts, which look like some kind of concatenation of sets by our previous comparison.

Difficulties arise when trying to write down a comparable symmetric group action in this setting, such as the issue of ordering. We were previously dealing with ordered sets and some equivalence relation on these, whereas now we have the potential to have both ordered sets, and the order in which the sets appear in the concatenation. We consider here an action on unordered strings of the equivalence classes of ordered sets in $\mathbb{Z}[O^n_m]/\sim$, where the action of Σ_n is exactly the same as that in Proposition 5.8.1.3 on each set in the collection. This makes sense from the point of view that there is no natural order on the parts within a partition.

We only need to go as far as considering an example for small n to see that there is no obvious extension of the action in this case. We immediately see this via the representation theory of the symmetric group.

Example 5.8.4.1 (The case $n = 4$)**.** We consider the action on sets for the case where $n = 4$ and $m = 2$ on partitions of shape $(2, 2)$, as this is the only remaining non-trivial partition shape that is not of the form previously considered.

The matrices of the Σ_4 action on the partitions $\{a_1, a_2\}\{a_3, a_4\}, \{a_1, a_3\}\{a_2, a_4\}, \{a_1, a_4\}\{a_2, a_3\}$ are the following.

$$(12) : \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad (23) : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (34) : \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

These have character -1, and by taking the necessary products, and comparing the characters with the Σ_4 character table, we get that the representation is $S^{(2,2)} \oplus S^{(1,1,1,1)}$. This is not a restriction of any Σ_5 representation, so the action does not extend to a hidden action on general partitions in this way.

Chapter 6

A cooperad of trees

In this chapter we will give a construction of a cooperad with cyclic structure that relates to topics covered in the other chapters of the thesis. In particular, the key ingredient of this cooperad will be the tree space from Chapter 5. This is closely related to a similar construction of Ching [Chi05], using weighted trees to give an operad equivalent to the Lie operad in the category of spectra.

It is known that the spectral Lie operad $\mathcal{L}$ is related to the Spanier-Whitehead dual of suitable suspensions of the partition complex Λ_n from section 5.4. This is a result of Arone and Mahowald in [AM99], and is also discussed by Heuts in [Heu20]. We know from Chapter 5 that Λ_n is homeomorphic to the tree space T_n. This motivates the construction of an operad using suspensions of the tree spaces T_n, in order to exploit the cyclic Σ_{n+1} action on these spaces to give an explicit cyclic structure on a suspension of the Lie operad in spectra.

Throughout this chapter, we will use both the non-skeletal and skeletal versions of the construction where one is more convenient than the other. We make use of the equivalences of definitions from chapters 3 and 4 to be able to do this.

The main results of this chapter are the introduction of *twisted cyclic structure* in Definition 6.4.0.1, the equivalence of the operad $\widehat{\mathcal{T}}$ we construct with a desuspension of the Lie operad in Theorem 6.7.0.4, and the anticyclic structure of $\widehat{\mathcal{T}}$ given in Theorem 6.8.0.1.

6.1 Background

We will need some definitions and properties of spectra and a category of spectra, before defining the Spanier-Whitehead dual. These results are quoted and can be found in more detail in [Mar83], [Spa56] and [HSS00].

Definition 6.1.0.1 (Smash product)**.** For X, Y based topological spaces, we define the smash product $X \wedge Y$ as

$$X \wedge Y := X \times Y/(X \vee Y).$$

This is associative and commutative.

Throughout this chapter, we will work with spectra in a suitable category. For our purposes, we only require finite spectra, and the Spanier-Whitehead category will suffice.

Definition 6.1.0.2 (Spectrum)**.** A *spectrum* is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of based topological spaces $X_n \in Top_*$, with structure maps

$$\sigma : S^1 \wedge X_n \to X_{n+1}, \qquad \text{for all } n \geq 0.$$

We note that there is a homeomorphism $S^1 \wedge X_n \cong \Sigma X_n$ between the above smash product and the reduced suspension of X_n.

We will now define the Spanier-Whitehead category, using the definition in [Mar83] that is also discussed in [Str20].

Definition 6.1.0.3 (Spanier-Whitehead category [Mar83, Part I])**.** We denote by $(\mathcal{SW}, \wedge, S^0)$ where $\wedge$ is the smash product and S^0 the unit, the category with objects $\Sigma^{\infty+n}X$ for n an integer, and morphisms maps f,

$$[\Sigma^{\infty+n}X, \Sigma^{\infty+m}Y] := \lim_{N\to\infty}[\Sigma^{N+n}X, \Sigma^{N+m}Y],$$

where we use the notation $[-,-]$ to mean based homotopy classes of maps.

The category $\mathcal{SW}$ is a symmetric monoidal category with the smash product $\wedge$.

We will also use the Spanier-Whitehead dual of an operad, and need a couple of its properties relating to homology and cyclic structure.

Definition 6.1.0.4 (Spanier-Whitehead dual)**.** Let X be a finite CW-complex. Then the Spanier-Whitehead dual $\mathcal{D} : \mathcal{SW} \to \mathcal{SW}$ is given by the map

$$\mathcal{D}X := F(X, S^0).$$

For P a symmetric sequence in $\mathcal{SW}$, we have

$$(\mathcal{D}P)(I) = \mathcal{D}(P(I)).$$

Then $\mathcal{D}P$ is the Spanier-Whitehead dual of P.

We have that for a cooperad P, the Spanier-Whitehead dual $\mathcal{D}P$ is an operad. We will use some properties of the Spanier-Whitehead dual including the interaction with the smash product and homology.

Lemma 6.1.0.5. *For $X, Y \in \mathcal{SW}$, we have*

$$[X \wedge Y, S^0] = [X, F(Y, S^0)].$$

Lemma 6.1.0.6. *For $X \in \mathcal{SW}$, then*

$$\widetilde{H}^{-k}(X) \cong \widetilde{H}_k(\mathcal{D}X).$$

6.2 Suspensions of the tree space

Recall the tree space $\widetilde{T}_n$ from Chapter 5. That is, the space of isomorphism classes of n-trees with leaves labelled by elements of $\underline{n}$ and internal edge lengths between 0 and 1.

Recall also the space T_n of fully grown n-trees, that is the subspace of $\widetilde{T}_n$ where at least one internal edge has length 1. As we discussed in Chapter 5, the space T_n is homotopy equivalent to a wedge of spheres,

$$T_n \simeq \bigvee_{(n-1)!} S^{n-3}.$$

We then have that the following quotient gives a suspension space,

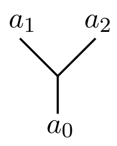
$$\widetilde{T}_n / T_n \cong ST_n.$$

This is because the space $\widetilde{T}_n$ is the cone on T_n, and so we can view the above quotient space as collapsing T_n to a point to give the unreduced suspension. The cone point in $\widetilde{T}_n$ is the corolla, that is, the tree with no internal edges, and we view this as $[(t, 1)]$ for $t \in T_n$. Here we are using $[-]$ to denote equivalence classes in the quotient space. We view T_n as the subspace $T_n \times \{0\}$ of $\widetilde{T}_n$, and quotient by this to obtain $ST_n = \widetilde{T}_n/T_n$. Therefore, this is a based space, with basepoint $[t]$ for t any fully grown tree.

We will use the notation S to mean unreduced suspension, and Σ to mean reduced suspension. In our example it will be useful to think of the reduced suspension ΣX for a based space X as the smash product $X \wedge S^1$ of the space with the circle.

Remark 6.2.0.1. For the construction that follows, we will switch to working with the non-skeletal tree spaces $\widetilde{T}_A$ for simplicity, mainly due to avoiding the complications added by the renumbering map in the skeletal case. We know by the equivalences of definitions in Chapters 2, 3 and 4 that we can do this.

Example 6.2.0.2 (T_A and ΣST_A for the two element based set $(A, a_0) = \{a_0, a_1, a_2\}$)**.** For the space with two non-basepoint points, there is only one tree t_A in T_A, which is the corolla, with no internal edges.



Since this tree is not fully grown, because it has no internal edges, the space $\widetilde{T}_A$ is empty. Therefore, the suspension ST_A is given by the space $T_A/\widetilde{T}_A$, which is the quotient of a one point space by the empty set. This gives the disjoint union of the set $\{t_A\}$ with a basepoint, and is therefore the space S^0.

Then taking the reduced suspension gives the space ΣST_A. We view this as the smash product $S^1 \wedge ST_A = S^1 \wedge S^0 = S^1$. The points in this space are equivalence classes $[(s, t_A)]$ where $0 \leq s \leq 1$, with the identification $[(0, t_A)] = [(1, t_A)]$.

6.3 Cooperad structure

We will give an explicit cooperad structure on the collection of spaces $\{\Sigma ST_n\}$ by defining cocomposition maps.

For this construction we will work with finite sets as labels for convenience and therefore define a non-skeletal cooperad, despite the tree space being defined skeletally. Due to the equivalences of skeletal and non-skeletal definitions in earlier chapters, we are able to do this. Therefore, we will work with spaces T_A of A-labelled trees.

We now define the main ingredient of the cooperad structure, namely the cocomposition map. Following this, we check that this is well-defined, continuous, and satisfies the required axioms for cooperad cocomposition.

Definition 6.3.0.1. Let $C = A \sqcup_a B$ for finite sets $A, B, C \in \mathcal{B}ij_*$ with $|A|, |B|, |C| \geq 2$ and $a \in A$. Let $t \in T_C$ be a tree with leaves labelled by the elements of C, and let t' be any tree in T_A, and t'' any tree in T_B. We denote by α both the name and the length of the internal edge of t that we cut when carrying out the following ungrafting operation.

We define a cocomposition map

$$\overline{\circ}_{A\sqcup_a B} : \Sigma ST_C \to \Sigma ST_A \wedge \Sigma ST_B,$$

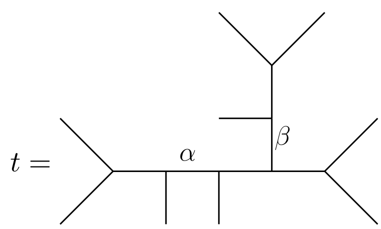
by

$$\overline{\circ}_{A\sqcup_a B} : [(s,t)] \mapsto \begin{cases} [(s,t',\alpha,t'')] & \text{if } t \text{ splits at edge } \alpha \text{ as } t' \text{ and } t'', \\ * & \text{else.} \end{cases}$$

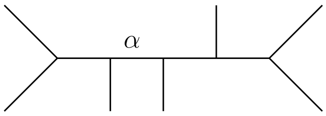
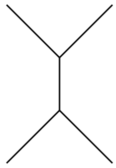
In the set of coordinates $[(s,t',\alpha,t'')]$, s and α are the reduced suspension coordinates that we view as the coordinate in a copy of S^1. Notice that α is the length of an internal edge in the tree t which means by definition that $0 \leq \alpha \leq 1$. Then we are at the basepoint when either of these coordinates is 0 or 1.

Remark 6.3.0.2. Notice that we use the notation $\overline{\circ}_{A\sqcup_a B}$ for the cocomposition map, rather than $\overline{\circ}_a$ as we have in earlier definitions. This is for clarity, and for ease of tracking the labelling sets of the different trees when ungrafting.

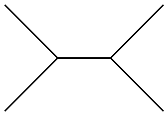
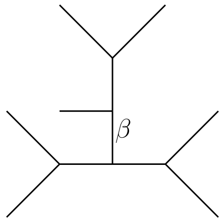
Example 6.3.0.3. Consider $[(s,t)]$ with $t = t_{A\sqcup_a B\sqcup_b C}$ the following tree. We will ungraft by cutting the edges α and β.



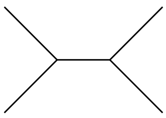
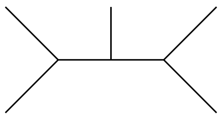
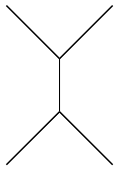
Ungrafting at edge β, we get $[(s, t_A \circ_a t_B, \beta, t_C)]$ with the trees as depicted below.

$t_A \circ_a t_B =$  , $t_C =$ 

Ungrafting at edge α, we get $[(s, t_A, \alpha, t_B \circ_b t_C)]$ with the trees as follows.

$t_A =$  , $t_B \circ_b t_C =$ 

Ungrafting the above tree $t_B \circ_b t_C$ at edge β, we get $[(s, t_A, \alpha, t_B, \beta, t_C)]$. We can see that we also obtain this by ungrafting first at edge β, and then the tree $t_A \circ_a t_B$ at edge α. The resulting trees are as follows.

$t_A =$  , $t_B =$  , $t_C =$ 

Proposition 6.3.0.4. *The cocomposition map*

$$\overline{\circ}_{A \sqcup_a B} : \Sigma ST_C \to \Sigma ST_A \wedge \Sigma ST_B$$

is well-defined and continuous.

Proof. **Well-defined**

Let $[(s, t)] \in \Sigma ST_C$, where $C = A \sqcup_a B$. We will check that if $[(s, t)] = *$, then

$$\circ_{A \sqcup_a B} : [(s, t)] \mapsto *.$$

Recall that we have $[(s, t)] = *$ if either the tree t is fully grown, or if the suspension coordinate s is either 1 or 0. In the case where s is 0 or 1, the map $\circ_{A \sqcup_a B}$ maps $[(s, t)]$ maps to $[(s, t', \alpha, t'')]$ which also contains the coordinate s, and so the basepoint in $\Sigma ST_A \wedge \Sigma ST_B$.

Then it remains to check that we also map fully grown trees to the basepoint. Fully grown trees are those with at least one internal edge of length 1. Then in cocomposing we either cut through a length 1 edge or an edge of length less than 1.

- If we cut through a length 1 internal edge, we have the coordinate $\alpha = 1$ in the reduced suspension coordinate S^1, which forces the right-hand side to be the basepoint.

- Finally, if we cut an internal edge that isn't length 1, again since the original tree is fully grown, at least one internal edge of length 1 remains in one of the resulting trees, so again we have the basepoint.

There is no other way for either tree t' or t'' to be fully grown, or for α to be either 0 or 1. Therefore, the map is well-defined.

Continuous

To check that it is continuous we consider continuous changes to the lengths of internal edges of $t \in T_C$.

- As an internal edge of t shrinks and its length approaches 0, if it is not the edge we cut then the corresponding edge in either t' or t'' also shrinks and so $\overline{\circ}$ behaves continuously with respect to such changes.

- If the edge we cut through shrinks and its length approaches 0, then the coordinate α approaches 0 which is the basepoint on the right-hand side. This is what we want, since the tree t is approaching a tree that no longer splits into trees t' and t'' and therefore maps to basepoint by definition.

- If an internal edge approaches 1 in a tree that is already fully grown, then we were already at the basepoint and mapping to the basepoint, so nothing changes.

- If t was not fully grown and the edge with length approaching 1 is not the one we cut then t is becoming fully grown and therefore the basepoint in ΣST_C. This means an edge in t' or t'' approaches length 1, giving a fully grown tree in either T_A or T_B which gives the basepoint as required.

- If t is not fully grown, and α is approaching 1, then this approaches the basepoint on the right, and t is approaching a fully grown tree which also gives the basepoint on the left.

- Any small change to an internal edge length in t other than this will result in either a small change in an internal edge of t' or t'', or a small change in α not resulting in anything being mapped to the basepoint.

Therefore, the map $\overline{\circ}_{A \sqcup_a B}$ is continuous. □

Remark 6.3.0.5. In order to have a counital cooperad structure, one would need a counit. The tree space is only defined for $|A| \geq 2$, and so T_W is empty. One could define ST_W as the one point space given by $\{w\}$, and the space ΣST_W to be S^0, given by $\{w\} \sqcup *$.

We could view cocomposition at this arity as ungrafting the tree t_w pictured above from a leaf of a tree t in ΣST_A, leaving t unchanged.

This would give a counit, and since S^0 is the unit of the category Top_*, would result in a reduced cooperad. Ching does a similar thing in his construction, however, we won't include this in ours because it doesn't give the correct sign when we consider cyclic action.

Theorem 6.3.0.6. *The cocomposition maps $\bar{\circ}$ are coassociative and equivariant. Therefore, this gives a non-skeletal non-counital cooperad structure in Top_* on the collection $\{\Sigma ST_A\}$.*

Proof. **Coassociativity**

For coassociativity we need that the diagrams for coassociativity from Definition 4.3.0.1 commute. Let A, B, $C \in \mathcal{B}ij_*$ be finite sets and let a, $a' \in A$, $b \in B$.

First, we have the case where we cut an edge α that separates t into trees in T_A and $T_{B\sqcup_b C}$, and the edge β separates t into trees in $T_{A\sqcup_a B}$ and T_C. This case is also depicted diagrammatically in Example 6.3.0.3.

$$\begin{array}{ccc}
\Sigma ST_A \wedge \Sigma ST_B \wedge \Sigma ST_C & \xleftarrow{\bar{\circ}_{A\sqcup_a B}\otimes id} & \Sigma ST_{A\sqcup_a B} \wedge \Sigma ST_C \\
{\scriptstyle id\otimes\bar{\circ}_{B\sqcup_b C}}\uparrow & & \uparrow{\scriptstyle \bar{\circ}_{(A\sqcup_a B)\sqcup_b C}} \\
\Sigma ST_A \otimes \Sigma ST_{B\sqcup_b C} & \xleftarrow[\bar{\circ}_{A\sqcup_a (B\sqcup_b C)}]{} & \Sigma ST_{A\sqcup_a B\sqcup_b C}
\end{array}$$

The other case is where we cut at an edge $\alpha \in t$ that splits t into trees in $T_{A\sqcup_{a'} C}$ and T_B, and also at an edge α' that separates t into trees in $T_{A\sqcup_a B}$ and T_C.

$$\begin{array}{ccc}
\Sigma ST_A \wedge \Sigma ST_B \wedge \Sigma ST_C & \xleftarrow{\bar{\circ}_{A\sqcup_a B}\otimes id} & \Sigma ST_{A\sqcup_a B} \wedge \Sigma ST_C \\
{\scriptstyle id\otimes t}\uparrow & & \\
\Sigma ST_A \wedge T_C \wedge \Sigma ST_B & & \uparrow{\scriptstyle \bar{\circ}_{(A\sqcup_a B)\sqcup_{a'} C}} \\
{\scriptstyle \bar{\circ}_{A\sqcup_{a'} C}\otimes id}\uparrow & & \\
\Sigma ST_{A\sqcup_{a'} C} \wedge \Sigma ST_B & \xleftarrow[\bar{\circ}_{(A\sqcup_{a'} C)\sqcup_a B}]{} & \Sigma ST_{A\sqcup_a B\sqcup_{a'} C}
\end{array}$$

Let $t \in T_{A\sqcup_a B\sqcup_b C}$ be a tree that splits into trees $t_A \in T_A$, $t_B \in T_B$, $t_C \in T_C$ by cutting edges α and β. The first case is the one where cutting α separates t_A from t, and cutting at β separates t_C from t. We wish to show that cutting the edge α and then the edge β is the same as cutting β first and then α. We have

$$\begin{aligned}
[(s,t)] &\overset{\text{cut }\alpha}{\mapsto} [(s, t_A, \alpha, t_B \circ_b t_C)] \\
&\overset{\text{cut }\beta}{\mapsto} [(s, t_A, \alpha, t_B, \beta, t_C)].
\end{aligned}$$

Alternatively, cutting at edge β first, we have

$$
\begin{aligned}
[(s,t)] &\overset{\text{cut}\,\beta}{\mapsto} [(s, t_A \circ_a t_B, \beta, t_C)] \\
&\overset{\text{cut}\,\alpha}{\mapsto} [(s, t_A, \alpha, t_B, \beta, t_C)].
\end{aligned}
$$

Therefore, the first of the diagrams commutes. The second case is a similar check, which we omit here.

Equivariance

Let $\sigma : A \to A', \tau : B \to B'$ be bijections of finite sets $A, A', B, B' \in \mathcal{B}ij_*$.

Then we need the following diagram to commute.

$$
\begin{array}{ccc}
\Sigma ST_{A'} \wedge \Sigma ST_{B'} & \xleftarrow{\;\bar{\circ}_{A' \sqcup_{\sigma(a)} B'}\;} & \Sigma ST_{A' \sqcup_{\sigma(a)} B'} \\
\Big\downarrow{\scriptstyle \sigma \otimes \tau} & & \Big\downarrow{\scriptstyle \sigma \circ_a \tau} \\
\Sigma ST_A \wedge \Sigma ST_B & \xleftarrow[\;\bar{\circ}_{A \sqcup_a B}\;]{} & \Sigma ST_{A \sqcup_a B}
\end{array}
$$

This clearly commutes since the operation $\bar{\circ}$ doesn't change the existing tree labels. When ungrafting a tree with labels $A \sqcup_a B$, one new label a is introduced to the tree in T_A. This is mapped by σ to $\sigma(a)$, which we expect from the diagram.

□

Therefore, we have a non-counital cooperad of trees, which we will denote by $\mathcal{T}$.

6.4 Twisted cyclic structure on $\mathcal{T}$

The properties of the tree space $\widetilde{T}_n$ and the Σ_{n+1} action on it are essential in proving the following result. We can easily permute the root label along with the other leaf labels, since our construction does not distinguish the root in the same way that other constructions do. We will briefly discuss the relationship with other constructions later.

We now introduce a modified version of a cyclic (co)operad called a *twisted cyclic* (co)operad. This is specifically for topological operads that are defined in based topological spaces with each space a suspension space.

This is because our cooperad $\mathcal{T}$ naturally has this structure. While we expect a cyclic structure due to the extended action of Σ_{n+1} on $\widetilde{T}_n$, the action on the suspension introduces a swap of suspension coordinates. This means the cooperad does not satisfy the conditions for a cyclic cooperad, but a special case that is very close.

In particular, this definition makes sense as a type of cyclic structure, because when we pass to the homology, the twist in suspension coordinates introduces signs. This gives precisely the conditions for an anticyclic operad in that case.

Definition 6.4.0.1 (Twisted cyclic operad)**.** Let P be a non-unital operad in Top_* such that each space $P(A)$ is a suspension $(\Sigma X)_A$ of a topological space. Then P has a twisted cyclic structure if for ΣX, $\Sigma Y \in Top_*$, $x \in X$, $y \in Y$, we have

1. $((s_1, p) \circ_x (s_2, q))\psi_{x,y} = (s_1, q(0, y)) \circ_y (s_2, p(0, x))$,
2. $((s_1, p) \circ_x (s_2, q))(0, x') = (s_1, p(0, x')) \circ_x (s_2, q))$.

That is, in the first case, where the root swaps between the two trees resulting from ungrafting, the suspension coordinates are swapped.

Remark 6.4.0.2. For a cooperad Q, the structure is defined analogously by reversing the direction of composition arrows in the diagrams to give cocomposition. The modified condition 2 is shown by the commutative diagram below. Let ΣX, ΣY be based spaces, $x \in X$, $y \in Y$.

$$\begin{array}{ccc}
\Sigma X \wedge \Sigma Y & \xleftarrow{\bar{\circ}_{X \sqcup_x Y}} & \Sigma(X \sqcup_x Y) \\
\downarrow{\scriptstyle (0,x)\otimes(0,y)} & & \downarrow{\scriptstyle \psi_{x,y}} \\
\Sigma X \wedge \Sigma Y & & \Sigma(Y \sqcup_y X) \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{\circ}_{Y \sqcup_y X}} \\
\Sigma Y \wedge \Sigma X & \xleftarrow[swap]{} & \Sigma Y \wedge \Sigma X
\end{array}$$

Theorem 6.4.0.3. *There is a twisted cyclic structure on the cooperad $\mathcal{T}$.*

Proof. In the tree space T_n, there is an action of Σ_{n+1} that permutes the leaf labels and root label as described in Section 5.3. This action is what gives a cyclic structure on the cooperad defined above.

We need to check that the cyclic conditions in Definition 4.3.0.1 for a cyclic cooperad are satisfied. Let $[(s,t)] \in \Sigma ST_{X \sqcup_x Y}$.

Twisted cyclic condition 1

The diagram for the second twisted cyclic condition is the following.

$$\begin{array}{ccc}
\Sigma ST_X \wedge \Sigma ST_Y & \xleftarrow{\bar{\circ}_{X \sqcup_x Y}} & \Sigma ST_{X \sqcup_x Y} \\
\downarrow{\scriptstyle (0,x)\otimes(0,y)} & & \downarrow{\scriptstyle \psi_{x,y}} \\
\Sigma ST_X \wedge \Sigma ST_Y & & \Sigma ST_{Y \sqcup_y X} \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \bar{\circ}_{Y \sqcup_y X}} \\
\Sigma ST_Y \wedge \Sigma ST_X & \xleftarrow[swap]{} & \Sigma ST_Y \wedge \Sigma ST_X
\end{array} \tag{6.1}$$

We have

$$\begin{aligned}
[(s,t)] &\xmapsto{\overline{\circ}_{X\sqcup xY}} [(s,t_X,\alpha,t_Y)] \\
&\xmapsto{(0,x)\otimes(0,y)} [(s,t_X(0,x),\alpha,t_Y(0,y))] \\
&\overset{t}{\mapsto} [(\alpha,t_Y(0,y),s,t_X(0,x))].
\end{aligned}$$

If instead we apply the map $\psi_{x,y}$ and then cocomposition, we have

$$\begin{aligned}
[(s,t)] &\xmapsto{\psi_{x,y}} [(s,t\psi_{x,y})] \\
&\xmapsto{\overline{\circ}_{Y\sqcup_y X}} [(s,t_Y(0,y),\alpha,t_X(0,x))] \\
&\xmapsto{swap} [(\alpha,t_Y(0,y),s,t_X(0,x))].
\end{aligned}$$

Therefore, diagram (6.1) commutes, and the second cyclic condition is satisfied.

Twisted cyclic condition 2

Finally, the diagram we require to commute for the third condition is the following.

$$\begin{array}{ccc}
\Sigma ST_X \wedge \Sigma ST_Y & \xleftarrow{\overline{\circ}_{X\sqcup_x Y}} & \Sigma ST_{X\sqcup_x Y} \\
\Big\downarrow{\scriptstyle (0,x')\otimes id} & & \Big\downarrow{\scriptstyle (0,x')} \\
\Sigma ST_X \wedge \Sigma ST_Y & \xleftarrow[\overline{\circ}_{X\sqcup_x Y}]{} & \Sigma ST_{X\sqcup_x Y}
\end{array} \tag{6.2}$$

We have

$$\begin{aligned}
[(s,t)] &\xmapsto{\overline{\circ}_{X\sqcup xY}} [(s,t_X,\alpha,t_Y)] \\
&\xmapsto{(0,x')\otimes id} [(s,t_X(0,x'),\alpha,t_Y)].
\end{aligned}$$

Applying the transposition $(0,x')$ first, we have

$$\begin{aligned}
[(s,t)] &\xmapsto{(0,x')} [(s,t(0,x'))] \\
&\xmapsto{\overline{\circ}_{Y\sqcup_y X}} [(s,t_X(0,x'),\alpha,t_Y)].
\end{aligned}$$

Therefore, diagram (6.2) commutes and the third twisted cyclic condition is satisfied. The cooperad given by Definition 6.3.0.1 does indeed have a twisted cyclic structure. □

6.5 Spanier-Whitehead dual of $\mathcal{T}$

We will now consider the Spanier-Whitehead dual of the cooperad $\mathcal{T}$. We will also pass back to the skeletal setting from this point, and work with the skeletal version of the operad $\mathcal{DT}$, which we know is equivalent from previous chapters. This is because it is easier to describe the homology in the skeletal setting and also the relationship of this operad with the Lie operad.

The Spanier-Whitehead dual of $\mathcal{T}$ is an operad $\mathcal{DT}$ in $\mathcal{SW}$. We have

$$\mathcal{DT} := F(\mathcal{T}, S^0),$$

with

$$(\mathcal{DT})(n) = \mathcal{D}(\mathcal{T}(n)).$$

6.6 Twisted cyclic structure on $\mathcal{DT}$

We wish to carry the twisted cyclic structure from $\mathcal{T}$ to the Spanier-Whitehead dual $\mathcal{DT}$ in spectra. The following Lemma tells us that $\mathcal{DT}$ is indeed twisted cyclic.

Lemma 6.6.0.1. *Let P be a twisted cyclic cooperad in $\mathcal{C}$. Then $\mathcal{D}P$ is a twisted cyclic operad in $\mathcal{SW}$.*

Proof. Let $\sigma : P(n) \to P(n)$ be the permutation in Σ_{n+1} given by the cyclic action of Σ_{n+1} on $P(n)$. Then we have in the operad $\mathcal{D}P = \{\mathcal{D}P(n)\}$, the map $\sigma^* \in \Sigma_{n+1}$

$$\mathcal{D}P(n) \xleftarrow{\sigma^*} \mathcal{D}P(n),$$

due to the fact that

$$[\mathcal{D}X_2, \mathcal{D}X_1] = [X_1, X_2].$$

Therefore, there is an action of Σ_{n+1} on $\mathcal{D}P(n)$ for all n, and the commutative diagrams defining the cyclic conditions for $\mathcal{D}P$ will be dual with the permutations inverted. These commute since $\Sigma \simeq \Sigma^{op}$. □

Corollary 6.6.0.2. *The operad $\mathcal{DT}$ in $\mathcal{SW}$ has a twisted cyclic operad structure.*

6.7 Homology of $\mathcal{T}$ and $\mathcal{DT}$

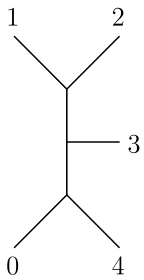
We know from Chapter 5 that the reduced (co)homology of the tree space is concentrated in dimension $n-3$, as it is a wedge of $(n-3)$-dimensional spheres. We have the following result of Robinson regarding cohomology of the tree space.

Proposition 6.7.0.1 ([Rob04, Theorem 4.1])**.** *There are isomorphisms of Σ_n-modules given by*

$$\widetilde{H}^*(T_n, \mathbb{Z}) \cong sgn \otimes Lie_n.$$

We will use the notation Lie_n to mean the Lie algebra on n symbols, and $\mathcal{L}ie(n)$ to mean the n^{th} arity of the Lie operad.

This can be seen by the natural correspondence between binary trees and Lie bracketings, for example the tree in the diagram below. These trees have three edges connected to each vertex, and correspond to the top dimensional simplices in the tree space.



$$\longleftrightarrow \quad [[[x_1, x_2]x_3]x_4].$$

We have the cycle $[c_n] \in \widetilde{H}^{n-3}(T_n)$ in cohomology given by the cyclically labelled trees, as discussed in Example 3.5.0.2. Then we have a basis

$$\{c_n\pi | \pi \in \Sigma_{n-1}\}.$$

Recall from section 2.6.0.7 that $\mathcal{L}ie(n)$ is given by linear combinations of all left bracketings of symbols $x_1, x_2, \ldots, x_n$ of the form

$$\alpha_n := [\ldots[[x_n, x_{n-1}]x_{n-2}]\ldots x_1],$$

without repetition. Then $\mathcal{L}ie(n)$ has a basis in cohomology

$$\{\alpha_n\pi | \pi \in \Sigma_{n-1}\}.$$

This is discussed in more detail by Robinson in the proof of the above proposition, and by Whitehouse in [Whi94].

Theorem 6.7.0.2. *The homology of the operad $\mathcal{DT}$ is the Lie operad in the category $\mathcal{SW}$ up to sign, for $n \geq 2$. That is,*

$$\{\widetilde{H}_*\mathcal{DT}\} = \{sgn \otimes \mathcal{L}ie(n)\}.$$

Proof. Let us consider the cohomology of the cooperad $\mathcal{T}$. The cohomology of the double suspension ΣST_n is concentrated in dimension $n-1$. We have

$$\widetilde{H}^*(\mathcal{T}) = \{\widetilde{H}^*(\Sigma ST_n)\} = sgn \otimes Lie_n.$$

After taking the dual, we have by Lemma 6.1.0.6 that the homology

$$\widetilde{H}_*(\mathcal{DT}) = sgn \otimes Lie_n$$

is concentrated in degree $n-1$.

Recall from section 4.7 that the cohomology of a topological cooperad is a cooperad in the category $Mod_{\mathbb{Z}}$. Then $\{\widetilde{H}_*(\mathcal{DT})\}$ has the structure of an operad. Composition is given by nesting Lie brackets.

We have the commutative diagram below.

$$\begin{array}{ccc}
sgn \cdot \mathcal{L}ie(m) \otimes sgn \cdot \mathcal{L}ie(n) & \xrightarrow{\bullet_m} & sgn \cdot \mathcal{L}ie(m+n-1) \\
\cong \uparrow & & \uparrow \cong \\
\widetilde{H}_{1-m}(\mathcal{D}\Sigma ST_m) \otimes \widetilde{H}_{1-n}(\mathcal{D}\Sigma ST_n) & \xrightarrow[\bullet_m]{} & \widetilde{H}_{2-m-n}(\mathcal{D}\Sigma ST_{m+n-1}),
\end{array}$$

Therefore, the two are equivalent as operads. □

Since we started with a non-unital cooperad, the above operad is also non-unital. We will define an operad $\widehat{\mathcal{T}}$ as follows.

Definition 6.7.0.3. We define the operad $\widehat{\mathcal{T}}$ by

$$\widehat{\mathcal{T}}(n) := \begin{cases} \widetilde{H}_{1-n}(\mathcal{D}\mathcal{T}) & \text{for } n \geq 2, \\ sgn_2 & \text{for } n = 1, \end{cases}$$

where sgn_2 is the sign representation of Σ_2. Then we have a graded Σ_{n+1}-module concentrated in degree $1-n$.

Theorem 6.7.0.4. *The operad $\widehat{\mathcal{T}}$ is the operadic desuspension of the Lie operad. That is,*

$$\{\widehat{\mathcal{T}}(n)\}_{n\geq 1} \cong \{\mathfrak{s}^{-1}\mathcal{L}ie(n)\}_{n\geq 1}.$$

Proof. This is automatic by comparing the result of Theorem 6.7.0.2 and the definition of $\widehat{\mathcal{T}}(1)$ with the definition of operadic desuspension given in Definition 2.9.0.1. □

6.8 Anticyclic structure on $\widehat{\mathcal{T}}$

Recall the result of Robinson and Whitehouse in [RW96], that was stated in Proposition 5.3.0.1, that the symmetric group Σ_{n+1} acts on the reduced homology groups $\widetilde{H}_{n-3}(T_n)$ with character

$$sgn \cdot (Ind_{\Sigma_n}^{\Sigma_{n+1}} Lie_n - Lie_{n+1}),$$

where sgn is the alternating character (the character of the sign representation). We also know that the restriction of this to the Σ_n action on $\widetilde{H}_{n-3}(T_n)$ has character $sgn \cdot \mathrm{Lie}_n$.

Recall from Definition 3.8.0.1 the anticyclic structure on an operad. Then we have the following result for the algebraic operad given by the homology.

Theorem 6.8.0.1. *The operad $\widehat{\mathcal{T}}$ has an anticyclic structure.*

Proof. The first anticyclic condition is true by definition of the operad $\widehat{\mathcal{T}}(1)$.

The diagram below results from taking the Spanier-Whitehead dual, and then homology of diagram (6.1). Therefore, this commutes and gives the second anticyclic condition.

$$\begin{array}{ccc}
\widetilde{H}_{1-m}(\mathcal{D}\Sigma ST_X) \otimes \widetilde{H}_{1-n}(\mathcal{D}\Sigma ST_Y) & \xrightarrow{\circ_{X\sqcup_x Y}} & \widetilde{H}_{2-m-n}(\mathcal{D}\Sigma ST_{X\sqcup_x Y}) \\
\Big\downarrow{\scriptstyle (0,x)\otimes(0,y)} & & \\
\widetilde{H}_{1-m}(\mathcal{D}\Sigma ST_X) \otimes \widetilde{H}_{1-n}(\mathcal{D}\Sigma ST_Y) & & \Big\downarrow{\scriptstyle -\psi_{x,y}} \\
\Big\downarrow{\scriptstyle t} & & \\
\widetilde{H}_{1-n}(\mathcal{D}\Sigma ST_Y) \otimes \widetilde{H}_{1-m}(\mathcal{D}\Sigma ST_X) & \xrightarrow[\circ_{Y\sqcup_y X}]{} & \widetilde{H}_{2-m-n}(\mathcal{D}\Sigma ST_{Y\sqcup_y X})
\end{array}$$

That is, the non-skeletal version of the second condition below, given in Definition 3.8.0.1.

$$(p \bullet_1 q)\tau_{m+n-1} = -(q\tau_n) \bullet_n (p\tau_m).$$

The swap in the suspension coordinates that occurs in the diagram (6.1) for $\mathcal{DT}$ results in the sign on the right-hand side after taking homology. This is equivalent to replacing the action of $\psi_{x,y}$ with $-\psi_{x,y}$ in the homology diagram.

The third condition is satisfied automatically by simply taking the dual and homology of diagram (6.2). □

Remark 6.8.0.2. We have a topological construction in $\mathcal{SW}$ that gives an anticyclic structure on the operad $sgn \otimes \mathcal{L}ie$. This is the desuspension of the Lie operad. We know from Getzler and Kapranov that taking operadic suspension of a cyclic operad gives an anticyclic operad, and so our construction shows that the spectral Lie operad has a cyclic structure.

6.9 Similar constructions

Finally, we will discuss some similar and related constructions elsewhere in the literature, and how they differ from the one here.

Operads and cooperads of trees are very common and occur in a number of different settings. There are also some examples of topological operads coming from spaces of trees similarly to this one. That is, trees with edges weighted by length, and with continuous shrinking of edges to pass between different trees in the space. We will discuss two of these examples here, namely the cooperad constructed by Ching in [Chi05], and the W-construction of Boardman and Vogt in [BV73]. In particular, Ching's construction is very closely related.

6.9.1 Boardman and Vogt's W-construction

Boardman and Vogt's W-construction [BV73] is similar to a bar construction on operads. It shares many similarities with what we do here, in particular it deals with a space of trees with weighted edges, and continuous shrinking of edges in the same way as we have in T_n. It is different because it gives weightings to both edges and vertices of the trees.

This construction produces an operad instead of a cooperad. This is the case elsewhere in the literature as well. Similar tree spaces are often used to construct operads, and indeed we could have constructed an operad with the tree spaces and grafting maps. However, if one starts with an operad, then the degrees in homology do not lead to the Lie operad. That is, the homology of the tree spaces is concentrated in the wrong degrees for it to give the operad composition structure of $\mathcal{L}ie$. This is why we begin with a cooperad and take the Spanier-Whitehead dual with suitable suspensions in this case.

6.9.2 Ching's trees

Ching's weightings on trees build in the homeomorphism from the tree space to partition space to his construction. In particular, weightings are given by distances to the root of the tree.

Ching talks about a reduced cooperad in Cat_+ formed from the categories $T(A)_+$. In this cooperad, the cocomposition maps are given by the ungrafting of trees. After taking the Spanier-Whitehead dual, the homology of the resulting operad is equivalent to the desuspension of the Lie operad.

Due to the nature of the weightings on the edges of the trees in this construction, it isn't possible to see a cyclic structure as it is in our construction.

6.9.3 Goodwillie derivatives of the identity

It is known that the spectral Lie operad arises in Goodwillie calculus, where it corresponds to the Goodwillie derivatives of the identity functor. This topic has been studied by Fresse, Ching, Arone in [Fre04], [AC11] and [Chi12b], Ching shows that it does indeed share the same homology up to sign, and it is true that the operad structure gives that of the Lie operad.

Proposition 6.9.3.1 ([Chi05, Example 5.6.8])**.** *The homology of the Goodwillie derivatives of the identity functor is given by*

$$\widetilde{H}_*(\partial_n I) = \begin{cases} Lie(n) \otimes sgn_n & \text{if } * = 1-n, \\ 0 & \text{otherwise.} \end{cases}$$

It has been shown that the Goodwillie derivatives of the identity correspond to the posets of partitions Λ_n, which are homeomorphic to the tree spaces as we discussed in Chapter 5. Therefore, the anticyclic structure we have constructed gives an anticyclic structure on homology of the Goodwillie derivatives of the identity.

Appendix A

A.1 Proof of Proposition 5.7.0.2

Proposition 5.7.0.2 The Σ_{n+1} action in definition 5.7.0.1 is well-defined.

First, we will prove the following claims.

Lemma A.1.0.1. *In the following equalities, all complements are in* $\underline{n}$*. Then we have*

1. $\overline{\sigma A} = \sigma\overline{A} \setminus \{0\} \cup \{\sigma(0)\}$ *for* $\sigma^{-1}(0) \notin A$,
2. $\overline{\sigma\overline{A}} = \sigma A \setminus \{0\} \cup \{\sigma(0)\}$ *for* $\sigma^{-1}(0) \in A$.

Proof.

1. $\sigma\overline{A} = \overline{\sigma A} \setminus \{\sigma(0)\} \cup \{0\}$, since $0 \in \sigma\overline{A}$ but $0 \notin \overline{\sigma A}$. Also, $\sigma(0) \in \overline{\sigma A}$ because $\sigma^{-1}(0) \notin A$, and $\sigma(0) \notin \sigma\overline{A}$ since $0 \notin \overline{A}$.
2. $0 \notin \overline{\sigma\overline{A}}$ again because $0 \notin \underline{n}$. Also, $\sigma(0) \notin \sigma A$ but $\sigma(0) \in \overline{\sigma\overline{A}}$ since $0 \notin \overline{A}$ and therefore $\sigma(0) \notin \sigma\overline{A}$.

□

Proof of Proposition 5.7.0.2. We need to check the identity and associativity conditions for a group action given in Definition 5.7.0.1.

The first of these is clear, whereas the second requires a case by case analysis. This is because the action is dependent on whether $\pi^{-1}(0)$ is in the m element subset of the partition. Let $\pi, \sigma \in \Sigma_{n+1}$.

Now, we start by considering $(\pi\sigma) \bullet A$.

- If $(\pi\sigma)^{-1}(0) \notin A \iff \pi^{-1}(0) \notin \sigma A$, then

$$(\pi\sigma) \bullet A = \pi\sigma A.$$

- If $(\pi\sigma)^{-1}(0) \in A \iff \pi^{-1}(0) \in \sigma A$, then

$$(\pi\sigma) \bullet A = \pi\sigma\overline{A} \cup \{\pi\sigma(0)\}.$$

Now, we consider $\pi \cdot (\sigma \bullet A)$.

1. If $\sigma^{-1}(0) \notin A$, then $\sigma \bullet A = \sigma A$. Then we have the two following cases.

- If $\pi^{-1}(0) \notin \sigma A$, then $(\pi\sigma)^{-1}(0) \notin A$, and we have the first case.

$$\pi \bullet (\sigma \bullet A) = \pi\sigma A.$$

- If $\pi^{-1}(0) \in \sigma A$, then $(\pi\sigma)^{-1}(0) \in A$, and this should agree with the second case.

$$\begin{aligned} \pi \bullet (\sigma \bullet A) &= \pi\overline{\sigma A} \cup \{\pi(0)\} \\ &= \pi(\sigma\overline{A} \setminus \{0\} \cup \{\sigma(0)\}) \cup \{\pi(0)\} \\ &= \pi\sigma\overline{A} \cup \{\pi\sigma(0)\}. \end{aligned}$$

2. If $\sigma^{-1}(0) \in A$, then $\sigma \bullet A = \sigma\overline{A} \cup \{\sigma(0)\}$. Then there are two further cases.

   - If $\pi^{-1}(0) \notin \sigma\overline{A} \cup \{\sigma(0)\}$, then $(\pi\sigma)^{-1}(0) \notin \overline{A} \cup \{0\}$ and therefore $(\pi\sigma)^{-1}(0) \in A$ and this should agree with the second case above.

$$\pi \bullet (\sigma \bullet A) = \pi\sigma\overline{A} \cup \{\pi\sigma(0)\}$$

   - If $\pi^{-1}(0) \in \sigma\overline{A} \cup \{\sigma(0)\}$, then $(\pi\sigma)^{-1}(0) \notin A$, and this is the same as the first case above.

$$\begin{aligned} \pi \bullet (\sigma \bullet A) &= \pi\overline{\sigma\overline{A} \cup \{\sigma(0)\}} \cup \{\pi(0)\} \\ &= \pi(\overline{\sigma\overline{A}} \cap \{\overline{\sigma(0)}\}) \cup \{\pi(0)\} \\ &= \pi(\sigma A \setminus \{0\}) \cup \{\pi(0)\} \\ &= \pi\sigma A. \end{aligned}$$

In all cases, the required associativity holds. □

A.2 Proof of Proposition 5.8.2.3

Proposition 5.8.2.3 The action of Σ_{n+1} on $\mathbb{Z}[O^n_m]/\sim$ in Definition 5.8.3.2 is well-defined.

Proof. The following braid relations need to be checked again:

$$\begin{aligned} t_n^2 &= id \\ t_{n-1}t_nt_{n-1} &= t_nt_{n-1}t_n \\ t_nt_i &= t_it_n, \text{ for } 1 \leq i \leq n-2. \end{aligned}$$

Again, this proof requires a case by case approach, this time depending on whether $n-1, n, i$, or $i+1$ are in $[A]$.

The first relation is clear in the case where $n \in A$. For $n \notin A$ we have

$$\begin{aligned} t_n^2[A] &= t_n([A] - \sum_{a \in A}[A^n_a]) \\ &= ([A] - \sum_{a \in A}[A^n_a]) - \sum_{a \in A} -[A^n_a] = [A], \end{aligned}$$

so the first of the relations holds. The second relation is again a case by case analysis. We will first state the following Lemma that will be needed in case 1.

Lemma A.2.0.1.

$$\sum_{a\in A}\sum_{a'\in A_a^{n-1}}[(A_a^{n-1})_{a'}^n] = \sum_{a\in A}[A_a^n].$$

Proof. We have that

$$\sum_{a\in A}[A_a^n] = \sum_{a\in A}[(A_a^{n-1})_{n-1}^n] \subset \sum_{a\in A}\sum_{a'\in A_a^{n-1}}[(A_a^{n-1})_{a'}^n],$$

So it remains to show that all other terms cancel.

In all the other terms we have the set A with two elements omitted, and replaced by $n-1$ and n. For each pair of elements in A, there is a set in the expansion for each way to substitute $n-1$ and n, and these pairs have opposite signs due to the equivalence relation. Therefore, they cancel as required. □

Then we have the following cases.

Case 1: $n, n-1 \notin A$.

$$\begin{aligned}
t_{n-1}t_nt_{n-1}[A] &= t_{n-1}\left([A] - \sum_{a\in A}[A_a^n]\right)\\
&= [A] - \sum_{a\in A}[A_a^{n-1}]\\
t_nt_{n-1}t_n[A] &= t_nt_{n-1}\left([A] - \sum_{a\in A}[A_a^n]\right)\\
&= t_n\left([A] - \sum_{a\in A}[A_a^{n-1}]\right)\\
&= [A] - \sum_{a\in A}[A_a^n] - \sum_{a\in A}\left([A_a^{n-1}] - \sum_{a'\in A_a^{n-1}}[(A_a^{n-1})_{a'}^n]\right)\\
&= [A] - \sum_{a\in A}[A_a^n] - \sum_{a\in A}[A_a^{n-1}] + \sum_{a\in A}\sum_{a'\in A_a^{n-1}}[(A_a^{n-1})_{a'}^n]\\
&= [A] - \sum_{a\in A}[A_a^{n-1}]
\end{aligned}$$

where the last equality is due to cancellation of terms in the sums.

Case 2: $n \notin A, n-1 \in A$.

$$\begin{aligned}
t_{n-1}t_nt_{n-1}[A] &= t_{n-1}t_n[A_{n-1}^n]\\
&= -t_{n-1}[A_{n-1}^n] = -[A]\\
t_nt_{n-1}t_n[A] &= t_nt_{n-1}\left([A] - \sum_{a\in A}[A_a^n]\right)\\
&= t_n\left([A_{n-1}^n] + \sum_{a\in A}[A_a^n] - [A_{n-1}^n] - [A]\right)\\
&= -\sum_{a\in A}[A_a^n] - [A] + \sum_{a\in A}[A_a^n] = -[A].
\end{aligned}$$

Case 3: $n \in A, n-1 \notin A$.

$$\begin{aligned}
t_{n-1}t_nt_{n-1}[A] &= t_{n-1}t_n[A_n^{n-1}] \\
&= t_{n-1}\left([A_n^{n-1}] - \sum_{a \in A_n^{n-1}} [(A_n^{n-1})_a^n]\right) \\
&= [A] + \sum_{a \in A_n^{n-1}} [(A_n^{n-1})_a^n] - [(A_n^{n-1})_{n-1}^n] - [A_n^{n-1}] \\
&= \sum_{a \in A_n^{n-1}} [(A_n^{n-1})_a^n] - [A_n^{n-1}] \\
t_nt_{n-1}t_n[A] &= -t_n[A_n^{n-1}] \\
&= -[A_n^{n-1}] + \sum_{a \in A_n^{n-1}} [(A_n^{n-1})_a^n].
\end{aligned}$$

Case 4: $n, n-1 \in A$.

$$t_{n-1}t_nt_{n-1}[A] = t_{n-1}[A] = -[A]$$
$$t_nt_{n-1}t_n[A] = t_n[A] = -[A]$$

For the final relation, we have $2^3 = 8$ cases, depending on whether each of $i, i+1, n$ are in the set A. In the cases where $i, i+1 \notin A$, the transposition t_i acts trivially, and clearly commutes with t_n. This leaves 6 cases remaining to check.

Case 1: $i \in A; i+1, n \notin A$

$$t_it_n[A] = t_i(-[A]) = -[A_i^{i+1}]$$
$$t_nt_i[A] = t_n[A_i^{i+1}] = -[A_i^{i+1}].$$

Case 2: $i+1 \in A; i, n \notin A$ (This is equivalent to Case 1)

$$t_it_n[A] = t_nt_i[A] = -[A_{i+1}^i].$$

Case 3: $i, n \in A; i+1 \notin A$

$$\begin{aligned}
t_it_n[A] &= t_i([A] - \sum_{a \in A}[A_a^n]) = [A_i^{i+1}] - \sum_{a \in A}[(A_a^n)_i^{i+1}] \\
&= [A_i^{i+1}] - \sum_{a \in A_i^{i+1}} [(A_i^{i+1})_a^n] \\
t_nt_i[A] &= t_n[A_i^{i+1}] = [A_i^{i+1}] - \sum_{a \in A_i^{i+1}} [(A_i^{i+1})_a^n].
\end{aligned}$$

Case 4: $i+1, n \in A; i \notin A$ (This is equivalent to Case 3)

$$t_it_n[A] = t_nt_i[A] = [A_{i+1}^i] - \sum_{a \in A_{i+1}^i} [(A_{i+1}^i)_a^n].$$

Case 5: $i, i+1 \in A; n \notin A$

$$\begin{aligned}
t_i t_n[A] &= t_i([A] - \sum_{a\in A}[A_j^n]) \\
&= -[A] + \sum_{a\in A}[A_a^n] - [A_i^n] - [A_{i+1}^n] - [(A_i^n)_{i+1}^i] - [(A_{i+1}^n)_i^{i+1}] \\
&= -[A] + \sum_{a\in A}[A_a^n] - [A_i^n] - [A_{i+1}^n] + [A_i^n] + [A_{i+1}^n] = -[A] + \sum_{a\in A}[A_a^n] \\
t_n t_i[A] &= t_n(-[A]) = -[A] + \sum_{a\in A}[A_a^n].
\end{aligned}$$

Case 6: $i, i+1, n \in A$

$$\begin{aligned}
t_i t_n[A] &= t_i(-[A]) = [A] \\
t_n t_i[A] &= t_n(-[A]) = [A].
\end{aligned}$$

So the relation holds in every case. □

Appendix B

Future work and questions

Configuration spaces

Another example of a hidden Σ_{n+1} action restricting to Σ_n action is the one that occurs in configuration spaces and is in fact related to the tree representation. Early and Reiner define this action in [ER19]. It can be described in terms of the Eulerian idempotents discussed in [Whi97], and it turns out that the Σ_{n+1} action on the top level of the graded cohomology ring corresponds to the tree representation.

As a topological space, $\mathrm{Conf}^n(X)$ is the complement of the *fat diagonal* in the product space X^n, and carries the subspace topology as a subspace of a cartesian product X^n with the product topology.

The extra Σ_{n+1} action can be seen by passing from configurations in $\mathbb{R}^3$ to configurations in a suitable quotient space of SU^2. That is, the extra point comes from adding a point at infinity to $\mathbb{R}^3$, to get a space homeomorphic to the sphere S^3. References for relevant results and properties of configuration spaces include [FH01], [Coh95] and [Knu18].

On the other hand, the Eulerian representations described in [Whi97] have lifts from Σ_n to Σ_{n+1}, which turns out to give the same representation as the one associated to the action in configuration spaces.

Question:

Can an explicit map from the tree spaces to configuration spaces (or vice versa) be constructed that illustrates how the extended symmetric group actions in the spaces are related and indeed give the same group representations?

Partition operads

A number of operads built from set partitions exist in the literature, for example those described in [Ebr+20]. This includes a non-symmetric operad via gap-insertion, and a coloured symmetric operad with block-substitution.

Additionally, we have seen in this thesis the relationship between trees and partitions. There are many well-known operads of trees, some that are known to have cyclic structure.

Questions:

- Can one of the extended symmetric group actions that we explored in Chapter 5 be used to show that a related operad is cyclic?
- Can we construct an operad or cooperad of partitions from the operads that are built from trees, and therefore get a cyclic structure in this way?

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Index of notation

Background

Operads

Cyclic operads

Cooperads

Trees and partitions

A cooperad of trees